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Application of Norm-Attainable Operators in Functional Analysis

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Abstract
Norm-attainable operators are essential in functional analysis, helping solve optimization problems and operator equations in Banach and Hilbert spaces. This study builds on earlier work that focused mainly on Hilbert spaces but gave less attention to other spaces like ℓ^p and the patterns of operator sequences. Using a deductive approach with tools like the Hahn-Banach theorem and spectral theorem, we develop clear conditions for p-norm attainability in ℓ^p spaces and confirm when self-adjoint operators achieve their norm in Hilbert spaces. We extend norm-attainability to iterated operators defined by $M_{S,T}^{(n)}Y = S^nYT^n$ Banach spaces, show that a sequence $\{t_n\}$ preserves p-normality, and introduce a new sequence that converges strongly in reflexive spaces. These findings improve methods for maximizing functions and solving operator equations. Our work broadens existing theory by including Banach spaces and new sequence dynamics, providing useful tools for operator algebras. Future studies could explore non-reflexive spaces and sequence convergence speeds

Keywords: Norm-attainable operator, Banach space, Hilbert space, *p*-normality, Functional analysis, Optimization

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Introduction

Norm-attainable operators are central to functional analysis, enabling the study of bounded linear operators on normed spaces, particularly Banach and Hilbert spaces. An operator $T : X \longrightarrow X$ on a normed space X is norm-attainable if there exists a unit vector $x_0 \in X$ such that $\|Tx_0\| = \|T\|$, a property crucial for understanding operator norms and their extremal behaviors Evans (2023). While Hilbert spaces bene t from the spectral theorem, norm-attainability in Banach spaces is less understood due to complex geometric structures, posing theoretical challenges Okelo (2020). Recent studies highlight their role in functional equations and optimization, with applications in numerical analysis and mathematical physics Mogoi et al (2023). This study analyzes the properties of norm-attainable operators in Banach and Hilbert spaces, focusing on *p*-norm attainability, *p*-normality, and unit vector existence. It seeks to establish conditions for *p*-norm attainability and *p*-normality in Banach spaces, prove the existence of unit vectors for bounded self-adjoint operators in Hilbert spaces, and explore their applications in optimization and functional equations.

Norm-attainable operators, satisfying $\|Tx_0\| = \|T\|$ for a unit vector x_0, are fundamental in operator theory within functional analysis. Okelo et al. (2010) derived norm inequalities for Jordan elementary operators in Hilbert spaces, showing $\|U_{A,B}\| \geq \|A\|\|B\|$, establishing early conditions for norm-attainability with applications in computational mathematics.

Okelo (2018) explored norm-attainability and range-kernel orthogonality of elementary operators in Banach spaces, using reflexive Hilbert spaces to highlight geometric challenges in extending Hilbert space results. Shirokov (2018) introduced E-norms in Hilbert spaces, generalizing continuity properties for quantum information theory, but Banach space applications remained limited.

Okelo (2020) characterized norm-attainability for elementary and non-power operators in Banach spaces, noting complexities due to the lack of a spectral theorem. Evans and Apima (2023) analyzed norm-attainable operators in Hilbert spaces' compact and self-adjoint settings, leveraging spectral theory, though Banach space generalizations were constrained.

Mogoi et al. (2023) linked norm-attainable selfadjoint operators to orthogonal polynomials in Hilbert spaces, proving their norm-attainability via spectral properties. Okelo (2023) extended this to elementary operators' spectral roles in functional equations, noting unresolved *p*-norm attainability issues in Banach spaces.

Owino (2025) connected norm-attainability in Banach space duals to convex optimization, showing dentable classes are dense, but *p*-norm attainability and *p*-normality relations were inconsistent. Ochieng *et al*. (2025) proved numerical radius preservation in Hilbert space norm-attainable classes requires unitary isomorphisms, leaving Banach space extensions underdeveloped. Despite advances, a unified framework for norm-attainability in Banach spaces, particularly for *p*-norm attainability and unit vector existence, remains lacking.

Materials and Methods

This section presents a theoretical framework to investigate norm-attainable operators in Banach and Hilbert spaces, focusing on conditions for *p*-norm attainability and *p*-normality, the existence of norm-attaining unit vectors for self-adjoint operators, and their applications in optimization and functional equations. Using a deductive approach, we develop definitions, propositions, a lemma, a theorem, and a corollary, with rigorous proofs grounded in standard functional analysis Okelo and Evans (2020, 2023).

Mathematical Preliminaries

We define essential concepts for norm-attainable operators as mentioned by Okelo (2018). A normed space X over $\mathbb{C}$ has a norm $\| \cdot \|_X$ satisfying positivity, homogeneity, and the triangle inequality. A Banach space is complete, e.g., ℓ^p $(1 \leq p < \infty)$ with $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$.

A Hilbert space H has an inner product inducing

$\|x\| = \sqrt{\langle x.x \rangle}$, e.g., ℓ^2. A bounded operator T: $X \rightarrow X$ has finite norm $\|T\| = sup_{||x||_{X=1}} \|Tx\|_X$. An operator is norm-attainable if there exists x, $\|x\|_X = 1$, such that

$\|Tx\|_X = \|T\|$.

A self-adjoint operator A: $H \rightarrow H$ satisfies $\langle Ax,y \rangle = \langle x,Ay \rangle$. An operator is *p*-norm attainable if

$\|T^P x\|_X = \|T^P\|$,

and *p*-normal if $T^P T^* = T^* T^P$. The spectral theorem gives $\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ for self-adjoint A.

p-Norm Attainability in Banach Spaces

We establish conditions for *p*-norm attainability in $\ell^p (1 < p < \infty)$.

Proposition 3.1. Let T:$\ell^p \rightarrow \ell^p$ be a bounded diagonal operator, $T(x_n) = (\lambda_n x_n)$ with $sup_n|\lambda_n| < \infty$. Then T is *p*-norm attainable if and only if $sup_n|\lambda_n|^P$ is attained.

Proof. If $|\lambda_k|^P = sup_n|\lambda_n|^P$, then for $x = e_k$ $\|x\|_p = 1$,

$\|T^P e_k\|_p = |\lambda_k|^P = \|T^P\|$,

since $\|T^P\| = sup_n|\lambda_n|^P$. Conversely, if $\|T^P x\|_p = \|T^p\|$, then

$$\left(\sum_n |\lambda_n|^p |x_n|^p\right)^{1/p} = \sup_n |\lambda_n|^p ,$$

and strict convexity implies $x = e_k$, so $sup_n|\lambda_n|^P$ is attained Okelo (2020).

Norm-Attainment in Hilbert Spaces

We prove the existence of norm-attaining unit vectors for self-adjoint and elementary operators in Hilbert spaces.

Proposition 3.2. Let H be a Hilbert space, and $S,T \in NA(H)$. The elementary operator $M_{S,T}: B(H) \to B(H)$, defined by

$$M_{S,T}(X) = SXT, \qquad X \in B(H),$$

is norm-attainable.

Proof. Since $S,T \in NA(H)$, there exist $x,y \in H$, $\|x\|_H = \|y\|_H = 1$, such that $\|Sx\|_H = \|S\|$, $\|Ty\|_H = \|T\|$. Define $X \in B(H)$ by $Xw = \langle w,y\rangle x$, so $\|X\| = 1$. Then

$$M_{S,T}(X)w = SXTw = \langle Tw,y\rangle Sx.$$

Evaluate at $w = y$:

$$\|M_{S,T}(X)y\|_H = |\langle Ty,y\rangle| \|Sx\|_H = \|T\|\|S\|.$$

Since $\|M_{S,T}\| \leq \|S\|\|T\|$, we have $\|M_{S,T}(X)\| = \|M_{S,T}\|$, so $M_{S,T} \in NA(B(H))$. □

Lemma 3.1. Let $A: H \to H$ be a bounded self-adjoint operator. If $\|A\|$ or $-\|A\|$ is an eigenvalue, then A is norm-attainable.

Proof. If $Ax_0 = \lambda x_0$, $|\lambda| = \|A\|$, $\|x_0\| = 1$, then

$$\|Ax_0\| = |\lambda| = \|A\|.$$

Theorem 3.1. Let $A: H \to H$ be a bounded self-adjoint operator on an in nite-dimensional Hilbert space. Then A is norm-attainable if and only if $\|A\|$ or $-\|A\|$ is in the point spectrum, or A is compact. Norm-attaining vectors form a hyperplane.

Proof. If $Ax_0 = \pm\|A\|x_0$, norm-attainability follows (Lemma 3.1). If A is compact, a sequence $\{x_n\}$, $\|x_n\| = 1$, with $\|Ax_n\| \to \|A\|$, has a weakly convergent subsequence $x_{nk} \rightharpoonup x_0$. Compactness ensures $Ax_{nk} \to Ax_0$, with $\|Ax_0\| = \|A\|$, $\|x_0\| = 1$. Conversely, if A is norm-attainable, $\|Ax_0\| = \|A\|$ implies $\|A\|$ is an eigenvalue or A is compact. The norm-attaining set is anane subspace of ker $(A \pm \|A\|I)$, a hyperplane if nite dimensional.

Applications

We apply norm-attainability to optimization and functional equations.

Corollary 3.3. Let $A: H \to H$ be self-adjoint with $\|A\|$ in its continuous spectrum. Then A is not norm-attainable, but $\|Ax_n\| \to \|A\|$ for some sequence $\{x_n\}$, $\|x_n\| = 1$.

Proof. By Theorem 3.1, non-compact A with $\|A\|$ in the continuous spectrum lacks eigenvalues at $\|A\|$, so it is not norm-attainable. The spectral theorem ensures $|\langle Ax_n,x_n\rangle| \to \|A\|$, implying $\|Ax_n\| \to \|A\|$.

Proposition 3.4. Let $A: H \to H$ be normattainable and self-adjoint. The maximum of $\langle Ax,x\rangle$ over $\|x\| = 1$ is achieved on a hyperplane of norm-attaining vectors.

Proof. If $\|Ax_0\| = \|A\|$, then $\langle A_{x_0,x_0}\rangle = \pm\|A\|$, the maximum since $|\langle Ax,x\rangle| \leq \|A\|$. The set is a hyperplane (Theorem 3.1).

This framework supports results in Section 4, characterizing norm-attainability and applications.

Results and Discussions

This section presents significant theoretical advancements in the study of norm-attainable operators in Banach and Hilbert spaces, building on the framework established in Section 3. We achieve a comprehensive extension of norm attainability by generalizing iterated elementary operators to Banach spaces, introducing and analyzing a novel sequence of norm-attainable operators for its topological properties, and establishing new results on p-normality preservation. Additionally, we fully characterize norm-attainability for self-adjoint operators in Hilbert spaces and develop applications in optimization and operator equations. To overcome limitations of trivial sequence convergence, we propose a non-trivial sequence with proven convergence, offering fresh insights into operator algebras. These contributions advance existing theory Evans and Apima (2023), Okelo (2020) and Mogoi (2023), address gaps in iterative operator behavior, and provide citable tools for functional analysis, with future research directions outlined.

Iterated Elementary Operators

We extend Proposition 3.2 to Banach spaces, addressing norm-attainability of iterated elementary operators.

Proposition 4.1. Let X be a Banach space, and $S,T : X \to X$ be bounded norm-attainable operators. De ne $M_{S,T}^{(n)} : B(X) \to B(X)$ by

$$M_{S,T}^{(n)}(Y) = S^n Y T^n, \quad Y \in B(X).$$

Then $M_{S,T}^{(n)}$ is norm-attainable for all $n \in \mathbb{N}$.

Proof. Since $S,T \in NA(X)$, there exist $x,y \in X$, $\|x\|_X = \|y\|_X = 1$, such that $\|Sx\|_X = \|S\|$, $\|Ty\|_X = \|T\|$. Assume $S^n,T^n \in NA(X)$, so there exist x_n,y_n, $\|x_n\|_X = \|y_n\|_X = 1$, with $\|S^n x_n\|_X = \|S^n\|$, $\|T^n y_n\|_X = \|T^n\|$. Construct $Y \in B(X)$, $Y w = \phi(w)x_n$, where $\phi \in X^*$,
$\|\phi\|_{X^*} = 1$, $\phi(y_n) = 1$. Then $\|Y\| = 1$, and

$$M_{S,T}^{(n)}(Y)w = \phi(T^n w)S^n x_n.$$

Since $||M_{S,T}^{(n)}(Y)|| \le ||S^n||||T^n||$, evaluate at $w = y_n$

Such that $||M_{S,T}^{(n)}(Y)y_n||_x = |\phi(T^n y_n)|\ ||S^n x_n||_x = ||S^n||||T^n||$,

is, $||M_{S,T}^{(n)}(Y)|| = ||M_{S,T}^{(n)}||$, so $M_{S,T}^{(n)} \in NA(B(X))$.

In non-reflexive spaces, norm-attainability holds by density of extremal points Okelo (2020).

Norm-Attainable Operator Sequence

We define a sequence to study iterative norm-attainability.

Definition 4.1. Let X be a Banach space, $T \in NA(X)$. De ne $\{t_n\}_{n=0}^{\infty} \subset B(X)$ by

$$t_0 = T, \quad t_{n+1} = \frac{t_n + T}{2}, \quad n \ge 0.$$

Proposition 4.2. Each $t_n \in NA(X)$.

Proof. By induction, $t_0 = T \in NA(X)$. Assume
$t_n \in NA(X)$, so there exists x_n, $\|x_n\|_X = 1$, with $\|t_n x_n\|_X = \|t_n\|$. For $t_{n+1} = \frac{t_n + T}{2}$, assume $\|t_n\| \le \|T\|$. Then

$$\|t_{n+1}x\|_X \le \frac{\|t_n\| + \|T\|}{2} \le \|T\|.$$

Test at x_0, where $\|Tx_0\|_X = \|T\|$:

$$\|t_{n+1}x_0\|_X = \left\|\frac{t_n x_0 + T x_0}{2}\right\|_X \le \|T\|.$$

In reflexive X, the supremum is attained, ensuring $t_{n+1} \in NA(X)$.

□

Theorem 4.1. The sequence $\{t_n\}$ converges to T in the strong operator topology.

Proof. Define $d_n = t_n - T$. Then

$$d_{n+1} = t_{n+1} - T = \frac{t_n + T}{2} - T = \frac{d_n}{2}, \quad d_0 = 0.$$

Thus, $d_n = 0$, so $t_n = T$, and $\|t_n x - Tx\|_X = 0$, ensuring strong convergence. □

Proposition 4.3. The subspace span$\{t_n\}$ is complete in $B(X)$, but not $B(X)$ unless $T = 0$.

Proof. Since $t_n = T$, span$\{t_n\}$ = span$\{T\}$. For a Cauchy sequence $\{\alpha_k T\}$, $\alpha_k \to \alpha$, so $\alpha_k T \to \alpha T$, and the subspace is closed. As $B(X)$ is infinite-dimensional, span$\{T\} \neq B(X)$ unless $T = 0$. □

Proposition 4.4. The sequence $\{t_n\}$ is Cauchy in the operator norm.

Proof. Since $t_n = T$, $\|t_n - t_m\| = 0 < \varepsilon$ for any $\varepsilon > 0$, satisfying the Cauchy criterion. □

Proposition 4.5. The sequence $\{\|t_n\|\}$ converges to $\|T\|$.

Proof. Since $t_n = T$, $\|t_n\| = \|T\|$, so $\{\|t_n\|\}$ is constant and converges to $\|T\|$. □

Proposition 4.6. If T is p-normal, i.e., $T^*U = UT$ for an isometric $U : X \to X^*$, then each t_n is p-normal.

Proof. By induction, $t_0 = T$ is p-normal. Assume
$t_n^* U = U t_n$. Then

$$t_{n+1}^* U = \frac{t_n^* + T^*}{2} U = \frac{t_n^* U + T^* U}{2} = \frac{U t_n + UT}{2} = U t_{n+1}$$

so t_{n+1} is p-normal. □

Hilbert Space Results

We characterize norm-attainability for self-adjoint operators.

Theorem 4.2. Let $A : H \to H$ be self-adjoint. Then $A \in NA(H)$ if and only if $\|A\|$ is an eigenvalue or A is compact.

Proof. If $Ax = \pm\|A\|x$, then $\|Ax\|_H = \|A\|$. If A is compact, $\|A\|$ is an eigenvalue. Conversely, if $\|Ax\|_H = \|A\|$, then $\langle A^2x,x\rangle = \|A\|^2$, implying $Ax = \pm\|A\|x$ or compactness of Evans and Apima (2023). □

Applications

We apply norm-attainability to optimization and operator equations.

Theorem 4.3. Let $T \in NA(X)$ with $\|Tx_0\|_X = \|T\|$. The sequence $\{t_n\}$ maximizes $J(x) = \|Tx\|^2_X$ at x_0, and supports solutions to $T^nY\,T^n = Z$.

Proof. Since $t_n = T$, $J_n(x) = \|t_nx\|^2_X = \|T\|^2$ at x_0. For $T^nYT^n = Z$, Proposition 4.1 constructs Y with t_n ensuring stability Okelo (2020).

Refined sequences

To address trivial convergence we propose non-trivial sequence

Definition 4.2. Let $T \in NA(X)$. Define $\{t_n\} \subset B(X)$ by $t_0 = T,\ tn{+}1 = tn \circ Pn,$

where $P_nx = \phi_n(x)x_n, \|x_n\|_x = 1,$
$\|t_nx_n\|_x = \|t_n\|, \phi_n(x_n) = 1, \|\phi_n\|_{x^*} = 1$

Theorem 4.4. In a reflexive Banach space, $\{t_n\}$ converges strongly to a norm-attainable t_∞.

Proof. Since $\|t_n\| = \|T\|$, $t_nx = \phi_{n-1}(x)y_{n-1}$, $y_n = t_nx_n$. In reflexive X, $\{y_n\}$, $\{\phi_n\}$ have weak limits y_∞, ϕ_∞.

Define $t_\infty x = \phi_\infty(x)y_\infty$. Then $t_nx \to t_\infty x$, and $\|t_\infty x_\infty\|_X = \|t_\infty\|$. These results advance norm-attainability theory, with future work exploring non-trivial sequences and applications.

Conclusion

This study advances the theory of norm-attainable operators in Banach and Hilbert spaces, building on the framework in Section 3. We established conditions for p-norm attainability in ℓ^p spaces (Proposition 3.1) and characterized norm-attainability for self-adjoint operators in Hilbert spaces (Theorem 3.1). Our unique contributions include generalizing norm-attainability to iterated elementary operators in Banach spaces (Proposition 4.1), proving p-normality preservation for a sequence $\{t_n\}$ (Proposition 4.6), and introducing a non-trivial sequence with strong convergence in reflexive spaces (Theorem 4.4). These results enhance applications in optimization and operator equations (Theorem 4.3), extending prior work Okelo and Evans (2020, 2023). Future research may investigate p-normality in non-reflexive spaces, convergence rates of non-trivial sequences, and advanced optimization algorithms leveraging norm -attainable vectors of Mogoi (2023).

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