Jewel Journal of Scientific Research (JJSR) 8(1&2): 13–22, 2023
©Federal University of Kashere-Nigeria (Print ISSN: 2384 – 6267, Online ISSN 2756-651X)
jjsr.fukashere.edu.ng

Integral tau Method for Certain Over-determined Fourth-Order Ordinary Differential Equations.

***[1]Ojo, V. O.; [2]Adeyefa, E. O. and [3]Faniyi, O. E.**
[1]Department of Statistics, Oyo State College of Agriculture and Technology, Igbo-Ora, Oyo State, Nigeria.
[2]Department of Mathematics, Federal University Oye Ekiti, Ekiti State Nigeria.
[3]Department of Mathematical Science, Anchor University, Ayobo, Ipaja, Lagos State, Nigeria.
***Corresponding Author**: ojovictoria2@yahoo.com ; +2348146864567

Abstract
This paper is concerned with the Integral tau methods for numerical solution of certain over determined Fourth-Order ordinary differential equations. Based on the degree of over determination and for the purpose of automation some recurrence relations were obtained. The automated Tau system obtained was tested on selected problem for the purpose of validation of the study. Numerical results further confirm that the order of the Tau approximant is also accurately estimated.

Keywords: Tau method, Variant, Formulation, Approximant, Perturbation term

Received: 26th June., 2023 *Accepted:* 27th Aug, 2023 *Published Online:* 27th Dec., 2023

Introduction
Lanczos proposed the tau method techniques in 1983 for the numerical solution of ordinary differential equation with some conditions given as

$$Ly_n(x)i = \sum_{r=0}^{m}\left(\sum_{k=0}^{N} P_{rk}\, x^k \right) y^{(r)}(x) = \sum_{r=0}^{n} f_r x^r \quad a \le x \le b \qquad (1.1)$$

$$L * Y(x_{rk}) = \sum_{r=0}^{m-!} a_{rk}\, y^{(r)}(x_{rk}) = \alpha_k \qquad k = 1(1)m \qquad (1.2)$$

By seeking an approximate solution of the form:

$$Yn(x) = \sum_{r=0}^{n} a_r x^r \qquad (1.3)$$

r$< +x$of $y(x)$ which is the exact solution of the corresponding perturbed system

$$L * Yn(x) = \sum_{r=0}^{n} f_r\, x^r Hn(x) \qquad (1.4)$$

$$LYn(x_{nk}) = \propto_k \text{ k= 1(1)m} \qquad (1.5)$$

Where

L is the linear differential operator $\propto_k,\ f_r,\ P_{nk,}\ \_N_M : r = 01(1)m. K = 0(1)N_r$ a *and* b *are real constants*, $y(r)$ *denoted the derivatives of order* r *of* $y(x)$. *The* perturbation term $H_n(x)$ in 1.4 is defined by

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} T_{n-m+i+1}(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{i=0}^{n-m+i+1} C_r{}^{(n-m+i+1)} x^r \qquad (1.6)$$

And $C_r{}^{(n)}s$are the coefficient of power of$x$ $(that\ is\ x^r)$ $in\ the\ nth$ degree chebyshev polynomial denoted and defined by

$$T_n(x) = \cos\left(ncos^{-1}\left[\frac{2x-a-b}{b-a}\right]\right) = \sum_{r=0}^{n} C_r{}^{(n)} \qquad (1.7)$$

The$r's$ are the free tau parameters to be determined alongside with $a_r$and $S$is the number of over-determination of (1.1), which is defined by

$$S = max[N_r - r : 0 \le r \le m, N_r \ge r] \ge 0 \qquad (1.8)$$

Literature Review

The Tau method was initially formulated as a tool for the approximation of special function of mathematical physics which could be expressed in terms of simple differential equations. It later developed into a powerful and accurate tool for the numerical solution of complex differential and functional equations. The main idea in it is to solve approximate problem. Accurate approximate polynomial solution in a linear ordinary differential equation with polynomial coefficient can be obtained by the Tau method introduced by Lanczos (1938). The method is related to the principle of economization of a differentiable function implicitly defined by a linear differential equation with polynomial coefficient. Techniques based on the Tau method have been reported in the literature with application to more general equations including non-linear ones as reported by Onumanyi and Ortiz (1982) also in the work of Adeniyi and Aliyu (2008), while techniques based on direct Chebyshev replacement have been discussed by Biala and Adeniyi (2015) and more recently in the work of Monsavi and Monsavi (2012). Further details on the Tau method can be found in references (Ojo and Adeniyi, 2012; Ojo and Adeniyi, 2012; Adeniyi, 1991; Aliyu, 2012; Isa and Adeniyi, 2013; Adeniyi and Aliyu, 2011; Yisa and Adeniyi, 2015; Adeniyi and Aliyu, 2007; Yisa and Adeniyi, 2012). Because of the limitation in some of the works reported by Aliyu (2012), this study seeks to extend the scope to fourth order problems with third degree over-determination.

The Integrated Formulation of the TAU method

Description of the integrated formulation

Let us consider the m-th order linear differential system

$$Ly(x) := \sum_{r=0}^{m} \alpha_r(x) y^{(}r)(x) = \sum_{r=0}^{f} f_r x^r, a \le x \le b \qquad (2.1)$$

$$L^* y(x_{rx}) := \sum_{r=0}^{m-1} \alpha_{rk} y^{(}r)(x_{rk}) = \alpha_k, \qquad k = 1(1)m \qquad (2.2)$$

Let $\iiint \overset{i}{\cdots} \int y(x)dx$ denote the indefinite integration i times applied to the function $g(x)$ and let

$$I_L = \iiint \overset{m}{\cdots} \int L(\cdot)dx \qquad (2.3)$$

The integral form of (2.3) now becomes

$$I_L(y(x)) = \iiint \overset{m}{\cdots} \int f(x)dx + C_{m-1}(x) \tag{2.4}$$

The tau approximant $y_n(x)$ of (3.1), satisfies the perturbed problem:

$$I_L(y_n(x)) = \iiint \overset{m}{\cdots} \int f(x)dx + H_{n+m+1}(x) \tag{2.5}$$

$$L^* y_n(x_{rk}) = \alpha_k, k = 1(1)m \tag{2.6}$$

where:

$$H_{n+m}(x) = \sum_{r=0}^{m+s+1} \tau_{m+s-r} T_{n-m+r+1}(x) \tag{2.7}$$

A class of Overdetermined Fourth Order Differential Equations

We consider here the integrated form of the tau method for the class of problems:

$$Ly(x) := \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = f(x), \qquad a \le x \le b \tag{3.1}$$

$$L^* y_n(x_{rk}) = \sum_{r=0}^{m-1} a_{rk} y_n^{(r)}(x_{rk}) = \alpha_k, \; k = 0(1)(m-1) \tag{3.2}$$

$$P_r(x) = \sum_{k=0}^{N_r} p_{rk} x^k \tag{3.3}$$

for the case $m = 4$ and $s = 3$.

So, we now derive a fifth degree approximants for the equation. From (3.1), the general case for $m = 4$, $s = 3$ is given by

$$\begin{aligned} Ly(x) \coloneqq (\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6 + \alpha_7 x^7) y^{iv}(x) + \beta_0 + \beta_1 x \\ + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4 + \beta_5 x^5 + \beta_6) y^{iii}(x) \\ + (\gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 + \gamma_5 x^5) y^{ii}(x) (\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 \\ + \lambda_4 x^4) y^{i}(x) \\ + (\mu_0 + \mu_1 x + \mu_2 x^2 \\ + \mu_3 x^3) y(x) = f(x) = \sum_{r=0}^{n} f_r x^r \end{aligned} \tag{3.4}$$

$$y(0) = \rho_0, y'(0) = \rho_1, y''(0) = \rho_2, y'''(0) = \rho_3$$

where, for convenience, we have chosen $\alpha, \beta, \gamma, \lambda$ and μ to denote $\rho_4, \rho_3, \rho_2, \rho_1$ and ρ_0 respectively; $x_{rk} = 0$ and $a = 0$. That is,

$$\begin{aligned} Ly(x) \coloneqq \int_0^x\int_0^u\int_0^t\int_0^s (\alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_3 v^3 + \alpha_4 v^4 + \alpha_5 v^5 + \alpha_6 v^6 + \alpha_7 v^7) y^{iv}(v) dvdsdtdu + \int_0^x\int_0^u\int_0^t\int_0^s \\ (\beta_0 + \beta_1 v + \beta_2 v^2 + \beta_3 v^3 + \beta_4 v^4 + \beta_5 v^5 + \beta_6) y'''(v) dvdsdtdu + \int_0^x\int_0^u\int_0^t\int_0^s (\gamma_0 + \gamma_1 v + \gamma_2 v^2 + \gamma_3 v^3 + \\ \gamma_4 v^4 + \gamma_5 v^5) y''(v) dvdsdtdu + \int_0^x\int_0^u\int_0^t\int_0^s (\lambda_0 + \lambda_1 v + \lambda_2 v^2 + \lambda_3 v^3 + \lambda_4 v^4) y'(v) dvdsdtdu + \\ \int_0^x\int_0^u\int_0^t\int_0^s (\mu_0 + \mu_1 v + \mu_2 v^2 + \mu_3 v^3) y(v) dvdsdtdu = \int_0^x\int_0^u\int_0^t\int_0^s f(V) dvdsdtdu + \tau_1 T_{n+7}(x) \\ + \tau_1 T_{n+6}(x) + \tau_2 T_{n+5}(x) + \tau_3 T_{n+} \quad \tau_4 T_{n+3}(x) + \tau_5 T_{n+2}(x) + \tau_6 T_{n+1}(x) \end{aligned} \tag{3.5}$$

Simplifying and equating the Corresponding Coefficient Powers of x, we have the recurrence relation:

$$\alpha_0 a_0 - \tau_1 C_0^{n+7} - \tau_2 C_0^{n+6} - \tau_3 C_0^{n+5} - \tau_4 C_0^{n+4} - \tau_5 C_0^{n+3} - \tau_6 C_0^{n+2} - \tau_7 C_0^{n+1} = \alpha_0 \rho_0 \quad (3.6)$$

$$\beta_0 a_0 + \alpha_0 a_1 - \tau_1 C_1^{n+7} - \tau_2 C_1^{n+6} - \tau_3 C_1^{n+5} - \tau_4 C_1^{n+4} - \tau_5 C_1^{n+3} - \tau_6 C_1^{n+2} - \tau_7 C_1^{n+1} = \alpha_1 \rho_1 - 2\alpha_0 \rho_1 - 2\beta_0 \rho_0 \quad (3.7)$$

$$\frac{1}{2}[(3\gamma_0 - 4\beta_1 - 7\alpha_2)a_0 - (\alpha_1 + \beta_0)a_1 - 2\alpha_0 a_0] - \tau_1 C_2^{n+7} - \tau_2 C_2^{n+6} - \tau_3 C_2^{n+5} - \tau_4 C_2^{n+4} - \tau_5 C_2^{n+3} - \tau_6 C_2^{n+2} - \tau_7 C_2^{n+1}$$

$$= \frac{\alpha_0 \rho_1 + 2\alpha_2 \beta_2 - \beta_0 \gamma_0 - 3\gamma_0 \rho_0 - 4\beta_1 \rho_0 - 7\alpha_2 \rho_0 + \alpha_1 \rho_1 + 2\alpha_0 \rho_2}{2} \quad (3.8)$$

$$\frac{1}{6}[(30\alpha_3 + 12\beta_2 + 5\gamma_1 + 2\lambda_0)a_0 + (8\alpha_2 + 5\beta_1 + 3\gamma_0)a_1 - 2(2\alpha_1 + \beta_0)a_2 + \alpha_0 a_3] - \tau_1 C_3^{n+7} - \tau_2 C_3^{n+6}$$
$$- \tau_3 C_3^{n+5} - \tau_4 C_3^{n+4} - \tau_5 C_3^{n+3} - \tau_6 C_3^{n+2} - \tau_7 C_3^{n+1}$$

$$= \frac{181\alpha_3 \rho_0 - 3\alpha_1 \rho_2 + 6\alpha_2 \rho_1 - 3\beta_0 \rho_2 + 10\beta_2 \rho_0 + 6\gamma_1 \rho_0 + 2\gamma_0 \rho_1 + \lambda_0 \rho_0 - 5\alpha_0 \rho_3 + 3\rho_1 + 5\beta_1 \rho_1}{6} \quad (3.9)$$

$$\frac{1}{24}[(116\alpha_4 + 30\beta_3 + 8\gamma_2 + \lambda_1 + \mu_0)a_0 + (42\alpha_3 + 14\beta_2 + 4\gamma_1 + 2\lambda_0)a_1 + (18\alpha_2 + 12\beta_1 + 6\gamma_0)a_2 - (18\alpha_1 + 6\beta_0)$$
$$a_3 + 24\alpha_0 a_4] - \tau_1 C_4^{n+7} - \tau_2 C_4^{n+6} - \tau_3 C_4^{n+5} - \tau_4 C_4^{n+4} - \tau_5 C_4^{n+3} - \tau_6 C_4^{n+2} - \tau_7 C_4^{n+1} = 0 \quad (3.10)$$

$$\frac{1}{120}[(720\alpha_5 + 168\beta_4 + 368\gamma_3 + 6\lambda_2 + \mu_1)a_0 + (264\alpha_4 + 96\beta_3 + 24\gamma_4 + 3\lambda_1 + \mu_0)a_1 + 2(42\alpha_3 + 34\beta_2 + 11\gamma_1$$
$$+ 2\lambda_0)a_2 + (60\alpha_2 + 42\beta_1 + 18\gamma_0)a_3 + 24(4\alpha_1 + \beta_0)a_4 + 120\alpha_0 a_5]$$
$$- \tau_1 C_5^{n+7} - \tau_2 C_5^{n+6} - \tau_3 C_5^{n+5} - \tau_4 C_5^{n+4} - \tau_5 C_5^{n+3} - \tau_6 C_5^{n+2} - \tau_7 C_5^{n+1} = 0 \quad (3.11)$$

$$\frac{1}{360}[(1440\alpha_6 + 900\beta_5 + 48\gamma_4 + 24\lambda_3 + \mu_2)a_0 + (1240\alpha_5 + 300\beta_4 + 60\gamma_3 + 8\lambda_2 + \mu_1)a_1 + (432\alpha_1 - 198\beta_3 -$$

$$46\gamma_2 - 5\lambda_1 + \mu_0)a_2 + (72\alpha_3 + 144\beta_2 + 42\gamma_1 + 6\lambda_0)a_3 + 120(11\alpha_2 - 8\beta_1 - 3\gamma_0)a_4 + 60(5\alpha_1 + \beta_0)a_5 +$$
$$360\alpha_0 a_6] - \tau_1 C_6^{n+7} - \tau_2 C_6^{n+6} - \tau_3 C_6^{n+5} - \tau_4 C_6^{n+4} - \tau_5 C_6^{n+3} - \tau_6 C_6^{n+2} - \tau_7 C_6^{n+1} = 0 \quad (3.12)$$

$$[\frac{[\alpha_1 k - \alpha_1 + \beta_0]}{k} a_{k-1} + \frac{[\alpha_2(k-1) - (2\alpha_2 + \beta_1)(k-1) - 3(2\alpha_2 + \beta_1 + \gamma_0)]}{k(k-1)} a_{k-2} + [\alpha_3(k-2)$$

$$\frac{(k-1)k - (3\alpha_3 + \beta_2)(k-2)(k-1) - 3(6\alpha_3 + 2\beta_2 + \gamma_1)(k-2) - 2(6\alpha_3 + 2\beta_2 + \gamma_1 - \lambda_0)]}{(k-2)(k-1)k} a_{k-3}$$

$$[\alpha_4(k-3)(k-2)(k-1)k - (4\alpha_4 + \beta_3)(k-3)(k-2)(k-1) - 3(12\alpha_4 + 3\beta_3 + \gamma_2)$$
$$\frac{(k-3)(k-2) - 2(24\alpha_4 + 6\beta_3 + 2\gamma_2 - \lambda_1)(k-3) + (24\alpha_4 + 6\beta_3 + 2\gamma_2 + \lambda_1 + \mu_0)]}{(k-3)(k-2)(k-1)k} a_{k-4}$$

$$[\alpha_5(k-3)(k-2)(k-1)k-(5\alpha_5+\beta_4)(k-3)(k-2)(k-1)-3(20\alpha_5+4\beta_4+\gamma_3)$$

$$\frac{(k-3)(k-2)-2(60\alpha_5+12\beta_4+3\gamma_3-\lambda_2)(k-3)-(120\alpha_5+24\beta_4+6\gamma_3+2\lambda_2+\mu_1)]}{(k-3)(k-2)(k-1)k}a_{k-5}$$

$$[\alpha_6(k-3)(k-2)(k-1)k-(6\alpha_6+\beta_5)(k-3)(k-2)(k-1)-3(30\alpha_6+5\beta_5+\gamma_4)$$

$$\frac{(k-3)(k-2)-2(120\alpha_6+120\beta_5+4\gamma_4+\lambda_3)(k-3)+(360\alpha_6+60\beta_5+12\gamma_4+3\lambda_3+\mu_2)]}{(k-3)(k-2)(k-1)k}a_{k-6}$$

$$[\alpha_7(k-3)(k-2)(k-1)k-(7\alpha_7+\beta_6)(k-3)(k-2)(k-1)-3(42\alpha_7+6\beta_6+\gamma_5)$$

$$\frac{(k-3)(k-2)-2(210\alpha_7+30\beta_6+5\gamma_5+\lambda_4)(k-3)+(840\alpha_7+120\beta_6+20\gamma_5+4\lambda_4+\mu_3)]}{(k-3)(k-2)(k-1)k}a_{k-7}$$

$$\tau_1C_K^{n+7}-\tau_2C_K^{n+6}-\tau_3C_K^{n+5}-\tau_4C_K^{n+4}-\tau_5C_K^{n+3}-\tau_6C_K^{n+2}-\tau_7C_K^{n+1}]$$

$$=\frac{f_{k-7}}{(k)(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)} \qquad (3.13)$$

$$\frac{1}{(n-2)(n-1)(n)(n+1)}[[(\alpha_7(n-2)[n^3-4n^2-120n-294]+840\alpha_7)-(\beta_6(n-2)[n^2+17n+62]-$$

$$120\beta_6)-(\gamma_5(n-2)(n-1)]-20\gamma_5)-(2\lambda_4(n-4)+\mu_3)a_{n-6}]+[(\alpha_6(n-2)(n^3-6n^2-85n-150)+360\alpha_6)$$

$$-(\beta_5(n-2)[n^2+14n+225]-60\beta_5)-\gamma_4(n-2)[3n+5]-12\gamma_4)-2\lambda_3(n-17)+\mu_2)]a_{n-5}+[(\alpha_5(n-2)$$

$$[n^3-5n^2-56n-60]-120\alpha_5)-(\beta_4(n-2)(n^2+11n+12)-24\beta_4)-\gamma_3(n-2)(3n+3)+6\gamma_3)+\lambda_2n+\mu_1)]$$

$$a_{n-4}+[\alpha_4(n-2)[-35n^3+20n^2+15n)+24\alpha_4)-(\beta_3(n-2)[n^2-10n-3]+6\beta_3)+(\gamma_2(n-2)(-3n-1)$$

$$+2\gamma_2)+(\lambda_1(5-2n)+\mu_0)]a_{n-3}+[(\alpha_3(n-2)(n^3-3n^2-19n+6)]+\beta_2[2n^2+10n)+\gamma_1(1-3n)-2\lambda_0)]$$

$$a_{n-2}+[(\alpha_2(n-2)(n^2-7n+6)]-\beta_1[n+3)-3\gamma_0]a_{n-1}+[\alpha_1(n-2)(n^2-n)(n+\beta_0)]a_n]$$

$$-\tau_1C_{n+1}^{n+7}-\tau_2C_{n+1}^{n+6}-\tau_3C_{n+1}^{n+5}-\tau_4C_{n+1}^{n+4}-\tau_5C_{n+1}^{n+3}-\tau_6C_{n+1}^{n+2}-\tau_7C_{n+1}^{n+1}]x^{n+1}$$

$$=\frac{f_{n-6}}{(n+1)(n)(n-1)(n-2)(n-3)(n-4)(n-5)} \qquad (3.14)$$

$$\frac{1}{(n-1)(n)(n+1)(n+2)}[[(\alpha_7(n-1)[n^3-4n^2-131n-420]+840\alpha_7)-(\beta_6(n-1)[n^2+19n+60]-$$

$$120\beta_6)-(\gamma_5(n-1)(3n+10)]-20\gamma_5)-(2\lambda_4(n-3)+\mu_3)a_{n-5}]+[(\alpha_6(n-1)(n^3-3n^2-86n-240)+$$

$$360\alpha_6)-(\beta_5(n-1)[n^2+16n+240)-60\beta_5)-\gamma_4(n-1)[3n+8]-12\gamma_4)-2\lambda_3(n+6)+\mu_2)]a_{n-4}+[(\alpha_5$$
$$(n-1)[n^3-2n^2-63n-120]-120\alpha_5)-(\beta_4(n-1)(n^2+13n+24)+24\beta_4)-\gamma_3(n-1)(3n+6)+6\gamma_3)-$$
$$2\lambda_2 n-\mu_1)]a_{n-3}+[\alpha_4(n+1)[n^3-39n^2+38n-48)-24\alpha_4)-(\beta_3(n-1)[n^2-2n)+6\beta_3)-(\gamma_2(n-1)$$
$$(3n+4)-2\gamma_2)+(\lambda_1(2n+3)+\mu_0)]a_{n-2}+[(\alpha_3(n-1)(n^3-19n)-12\alpha_3)-\beta_2(n^2-7n)-4\beta_2)$$
$$(\gamma_1(3n+2)-2\lambda_0)]a_{n-1}+[(\alpha_2(n-1)(n^2-7n)-\beta_1[n+4)-3\gamma_0]a_n-\tau_1C_{n+2}^{n+7}-\tau_2C_{n+2}^{n+6}-\tau_3C_{n+2}^{n+5}$$
$$-\tau_4C_{n+2}^{n+4}-\tau_5C_{n+2}^{n+3}-\tau_6C_{n+2}^{n+2}=\frac{f_{n-5}}{(n+2)(n+1)(n)(n-1)(n-2)(n-3)(n-4)} \qquad (3.15)$$

$$\frac{1}{(n)(n+1)(n+2)(n+3)}[[(\alpha_7 n[n^3-n^2-136n-554)+840\alpha_7)-(\beta_6 n)[n^2+21n+80]-$$
$$120\beta_6)-(\gamma_5 n(3n-7)]-20\gamma_5)-2\lambda_4(n-2)+\mu_3)]a_{n-4}]+[(\alpha_6 n(n^3-85n-336)+$$
$$360\alpha_6)-(\beta_5 n(n^2+18n+25)-60\beta_5)-(\gamma_4 n(3n-5)-12\gamma_4)-2\lambda_3(n-15)+\mu_2)]a_{n-3}+[(\alpha_5 n$$
$$[n^3+n^2-64n-184]-120\alpha_5)-(\beta_4 n(n^2+15n+38)+24\beta_4)-\gamma_3 n(3n+9)+6\gamma_3)-2\lambda_2(n+1)-\mu_1)]$$
$$a_{n-2}+[\alpha_4 n[n^3+2n^2-37n-86)+24\alpha_4)-(\beta_3 n(n^2+12n+23)-(\gamma_2 n(3n+7)-2\gamma_2)$$
$$-(\lambda_1(2n-1)+\mu_0)]a_{n-1}+[(\alpha_3 n(n^3+3n^2-16n-30)-\beta_2 n(n+9)-12\beta_2)-(\gamma_1(3n+5)-2\lambda_0)]a_n$$
$$-\tau_1C_{n+3}^{n+7}-\tau_2C_{n+3}^{n+6}-\tau_3C_{n+3}^{n+5}-\tau_4C_{n+3}^{n+4}-\tau_5C_{n+3}^{n+3}$$
$$=\frac{f_{n-4}}{(n+3)(n+2)(n+1)(n)(n-1)(n-2)(n-3)} \qquad (3.16)$$

$$\frac{1}{(n+1)(n+2)(n+3)(n+4)}[[(\alpha_7(n+1)[n^3+2n^2-135n-688]+840\alpha_7)-(\beta_6(n+1)(n^2+23n+102)$$
$$-120\beta_6)-(\gamma_5(n+1)(3n+16)]-20\gamma_5)-(2\lambda_4(n-1)+\mu_3)a_{n-3}]+[(\alpha_6(n+1)(n^3-9n^2-94n-282)+$$
$$360\alpha_6)-(\beta_5(n+1)(n^2+20n+276)-60\beta_5)-\gamma_4(n+1)[3n+14]-12\gamma_4)-2\lambda_3(n-14)+\mu_2)]a_{n-2}+$$
$$[(\alpha_5(n+1)[n^3+4n^2-59n-234]-120\alpha_5)-(\beta_4(n+1)(n^2+17n+54)+24\beta_4)-\gamma_3(n+1)(3n+36)+6\gamma_3)$$
$$-2\lambda_2(n+2)-\mu_1)]a_{n-1}+[\alpha_4(n+1)[n^3+5n^2-30n-120)+24\alpha_4)-(\beta_3(n+1)[n^2-4n-24)-6\beta_3)-$$
$$(\gamma_2(n+1)(3n+2)-2\gamma_2)-(\lambda_1(2n+1)+\mu_0)]a_n-\tau_1C_{n+4}^{n+7}-\tau_2C_{n+4}^{n+6}-\tau_3C_{n+4}^{n+5}-\tau_4C_{n+4}^{n+4}$$
$$=\frac{18}{(n+4)(n+3)(n+2)(n+1)(n)(n-1)(n-2)} \qquad (3.17)$$

$$\frac{1}{(n+2)(n+3)(n+4)(n+5)}[[(\alpha_7(n+2)[n^3+5n^2-128n-822]+840\alpha_7)-(\beta_6(n+2)(n^2+25n+12)$$
$$-120\beta_6)-(\gamma_5(n+2)(3n+19)]-20\gamma_5)-(2\lambda_4 n+\mu_3)]a_{n-2}+[(\alpha_6(n+2)(n^3+6n^2-85n+520)+$$

$$360\alpha_6) - (\beta_5(n+2)(n^2+22n+297) - 60\beta_5) - \gamma_4(n+2)(3n-5) - 12\gamma_4) - 2\lambda_3(n-13) + \mu_2)]a_{n-1} +$$
$$[(\alpha_5(n+2)[n^3+7n^2-48n-300] - 120\alpha_5) - (\beta_4(n+2)(n^2+19n+72) + 24\beta_4) - \gamma_3(n+2)(3n+15) + 6\gamma_3)$$
$$-2\lambda_2(n+3) - \mu_1)]a_n - \tau_1 C_{n+5}^{n+7} - \tau_2 C_{n+5}^{n+6} - \tau_3 C_{n+5}^{n+5} = \frac{f_{n-2}}{(n+5)(n+4)(n+3)(n+2)(n+1)(n)(n-1)} \quad (3.18)$$

$$\frac{1}{(n+3)(n+4)(n+5)(n+6)}[[(\alpha_7(n+3)[n^3+8n^2-11n-944] + 840\alpha_7) - (\beta_6(n+3)(n^2+27n+152)$$
$$-120\beta_6) - (\gamma_5(n+3)(n+14)] - 20\gamma_5) - (2\lambda_4(n+1) + \mu_3)]a_{n-1} + [(\alpha_6(n+3)(n^3+9n^2+70n-600) +$$
$$360\alpha_6) - (\beta_5(n+3)(n^2+24n+320) - 60\beta_5) - \gamma_4(n+3)(3n+20) - 12\gamma_4) - 2\lambda_3(n-12) + \mu_2)]a_n +$$
$$-\tau_1 C_{n+6}^{n+7} - \tau_2 C_{n+6}^{n+6} = \frac{f_{n-1}}{(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)(n)} \quad (3.19)$$

$$\frac{1}{(n+4)(n+5)(n+6)(n+7)}[[(\alpha_7(n+4)[n^3+11n^2+58n-630] + 840\alpha_7) - (\beta_6(n+4)$$
$$(n^2+29n+180)120\beta_6) - (\gamma_5(n+4)(3n+25) - 20\gamma_5) - (2\lambda_4(n+2) + \mu_3)]a_n$$
$$-\tau_1 C_{n+7}^{n+7} = \frac{f_n}{(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)} \quad (3.20)$$

A Numerical Experiment

We consider here the following problems for experiment with our results of the preceding sections. The exact error is defined by $\varepsilon *= \max_{0\le x\le 1}[|Y(x_k) - Y_n(x_k)|], 0 \le x \le 1, [x_k] = [0.01k], k = 0(1)100$

Example 4.1

$$LY(x) := y^{iv}(x) - y''(x) + \left(\frac{3}{2} + \frac{1}{2}x + \frac{1}{4}x^2\right)y(x) = \frac{9}{4} + \frac{3}{2}x + x^2 + \frac{3}{8}x^3$$

$y(0) = \frac{3}{2}, y'(0) = \frac{1}{2}, y''(0) = \frac{1}{2}, y'''(0) = \frac{1}{2}$

$m = 4, \quad S = 2$

Table 4.1a : The linear equations obtained we[...]ved by maple package as shown below(case n=6)

x $\boldsymbol{x}$	Exact	Approximate	Error
0	1.500000000000000	1.500000000000000	0.0000E+00
0.1	1.552585459000000	1.552585459051886	0.0000E-00
0.2	1.610701379000000	1.610701380771994	1.0000E-9

0.3	1.674929403000000	1.674929430790961	2.7000E-8
0.4	1.745912348000000	1.745912536389091	1.8800E-7
0.5	1.824360635000000	1.824361457272781	8.2200E-7
0.6	1.911059400000000	1.911062077174601	2.6770E-6
0.7	2.006876354000000	2.006883414463614	7.0600E-6
0.8	2.112770464000000	2.112786300940194	1.5836E-5
0.9	2.229801556000000	2.229832621418836	3.1065E-5
1.0	2.359140914000000	2.35919491363453	5.39990E-5

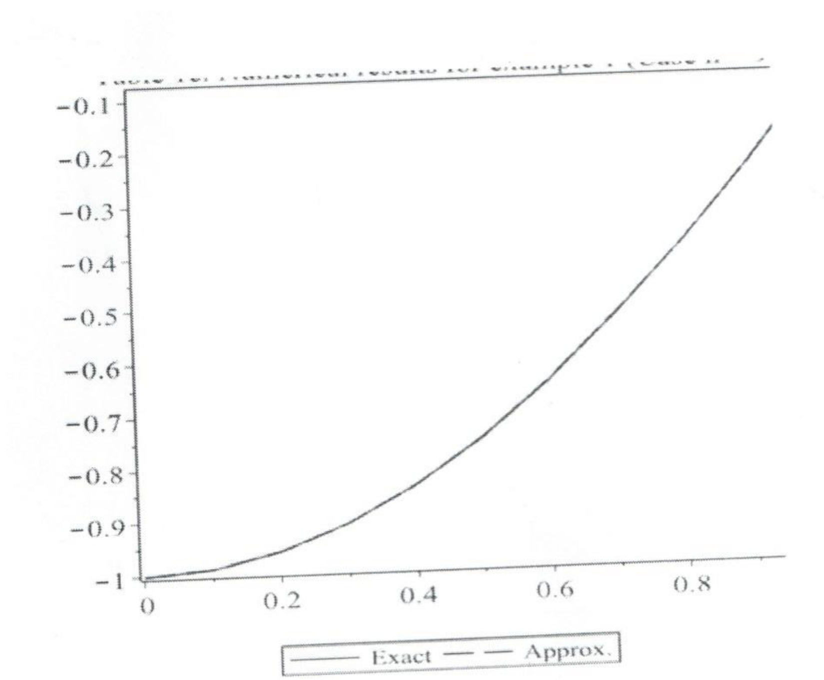

Figure 4.1a. Graphical representation of numerical result for Example 4.1 (Case n=6)

Table 4.1b: The numerical results for problem 4.1b (case n=7)

X	Exact	Approximate	Error
0	1.500000000000000	1.500000000000000	0.00000000
s0.1	1.552585459000000	1.552585459051886	0.00000000
0.2	1.610701379000000	1.610771380771994	-7.0001E-5
0.3	1.674929403000000	1.674929430790960	-2.7000E-8
0.4	1.745912348000000	1.745912536389081	-1.8800E-7
0.5	1.824360635000000	1.824361457272606	-8.2200E-7
0.6	1.911059400000000	1.911062077192740	-2.6770E-6
0.7	2.006876354000000	2.006883414449804	-7.0600E-6
0.8	2.112770464000000	2.112786300861840	-1.5836E-5
0.9	2.229801556000000	2.229832621056559	-3.1065E-5
1.0	2.359140914000000	2.359194939938212	-5.4025E-5

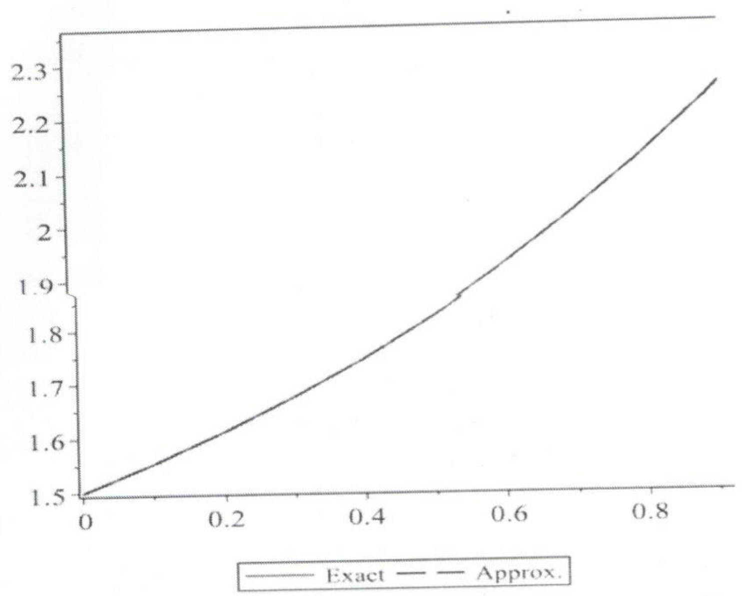

Figure 4.1b. Graphical representation of numerical results for numerical result for Example 4.1 (Case n=7)

Discussion of Results

From the tau approximation problem of this study, the numerical results presented in tables 4.1a-4.1b show the closeness of tau approximant to the desired analytic solutions. This shows that the proposed method is quite effective for handling the class of problems addressed.

Conclusion

The derivation of an approximation scheme for a fourth order ordinary differential equations with third degree overdetermination by the integral tau method has been presented. The class of ordinary differential equations under consideration were integrated four times and then pertubing the resulting equations. This is to guaranttee an improved accuracy of the desired approximation viz-a-viz those of the recursive and the differential formulations .Adeniyi (1991). Numerical evidences obtained from the selected problem show that the method is accurate and effective.

Declaration

Competing Interest: There is no competing interest in the work.

Funding: The work was fully funded by the authors

References

Lanczos, C. (1938): Trigonometric Interpolation of Empirical and Analytical Function. *J. Maths. Physics* 17: 123-199.

Onumanyi, P. and Ortiz, E.L (1982). Numerical Solution of Higher Order Boundary Value Problems for Ordinary Differnetial Equations with an Estimation of the Error. *Intern. J. Numer. Mech. Engrg,* 18: 775-781.

Adeniyi,R.B and Aliyu,I.M (2008).On Tau Method for the Class of non-overdetermined second order differential equations. *J. Niger. Assoc.Math. Physc.*12: 387-398.

Ojo, V.O and Adeniyi, R.B. (2012). The Differential Form of the Tau Method and its Error Estimate for third order Non- Overdetermined differential Equations.*Gen.Math. Notes* 11(1): 41-49.

Ojo, V.O. and Adeniyi R.B (2012).The Integrated Formulation of the Tau Method and its Error Estimate for third order Non- Overdetermined Differential Equations. *IJERT* 1(9): 1-12.

Biala,T.A and Adeniyi, R.B(2015). A Linear Tau Collocation Method for Partial Differential Equations. *Journal of the Nigerian Association of Mathematical Physics.30:41-48.*

Mousavi, S. and Mousavi, W. (2012) New Application of Tau Method for Solving Differential Equations of the lan Emden type. *World Applied Science Journal* 18(5): 642-644.

Adeniyi, R.B. (1991). On The Method of Tau Method for Numerical Solution of Ordinary Differential Equations. *Doctoral Thesis, University of Ilorin, Ilorin, Nigeria. pages* 122pp.

Aliyu, I. M. A. (2012). Computational Error Estimate of Integrated Formulation of the Tau Method for some Class of ODEs. *Doctoral Thesis, University of Ilorin, Ilorin, Nigeria.* 217pp

Issa, K and Adeniyi, R.B. (2013). A Generalised Scheme for the Numerical Solution of Initial Value Problems in ODEs by the Recursive Formulation of the Tau Method. *International Journal of Pure and Applied Mathematics.* 88: 1-13.

Adeniyi, R.B. and Aliyu, A.I. (2011). On the Integrated Formulation of the Tau Method involving at most Two Parameters. *International Journal on Research and Development* 3: 104-114.

Yisa, B.M and Adeniyi, R.B. (2015). On the Generalisation of the Error and the Error Estimation Process of the Ortizs Recirsive Formulation of the Tau Method. *Journal of The Nigerian Mathematical Society*. 34: 70-82.

Adeniyi, R.B. and Aliyu, A.I.M (2007). On the integrated formulation of numerical schemes for a class of Non-overdetermined second order Differential Equations. *Journal of the Nigerian Mathematical Society, 29:111-128.*

Yisa, B. M. and Adeniyi, R. B.(2012). On the variants of the tau method for solution of IVPs in first order differencial equations. *Kenya Journal of Sciences*. 15: 1-12.