Jewel Journal of Scientific Research (JJSR) 10(2): 293–316, 2025
©Federal University of Kashere-Nigeria (Print ISSN: 2384 – 6267, Online ISSN 2756-651X)
https://journals.fukashere.edu.ng/index.php/jjsr

Counting Death to Measure Survival: An Analytic Technique of Measuring Mortality Rate Intensity from the Heligman-Pollard's Law

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Abstract
The Heligman-Pollard model is distinguished to be one of the most influential multi-component parsimonious laws of mortality incorporating age-specific mortality trends with interpretable framework. However, the complexity of its form, computation challenges and lack of mechanistic grounding being overlooked in actuarial literature serve as the motivation in addressing it more as an analytic law. The objectives are to generate mortality table, derive the probability of survival function and test its asymptotic properties, and state the structural properties of the law. Computational evidence from our results reveals that for both sexes, the second component exhibits lognormal kernel expressed as the accidental hump into higher adulthood within the age interval $10 \le x \le 50$ for both sexes. Consequently, the lognormal behaviour caused its trajectories to exhibit leptokurtic curves which described the shape of the mortality distribution in the second terms. The implication is that the leptokurtic mortality distribution has a higher number of intensities clustered around the mean deaths with more extreme severities in the tails relative to a normal distribution.

Keywords: Kernel, leptokurtic, parsimonious, severities, survival
Received: 7th Sept, 2025 *Accepted:* 21th Nov, 2025 *Published Online:* 26th Dec, 2025

Introduction

The Heligman–Pollard mortality function was developed in Heligman and Pollard (1980) in form of a consolidated eight-parameter parsimonious mortality law to characterise the age pattern of death over the entire human life span. It delineates the probability of death at different ages x expressed as odd ratio $\frac{q_x}{p_x}$ into three different analytic terms: an infant mortality, accident hump characterising high death risk in young adulthood and an exponential old-age mortality component in form of Gompertz's senescent mortality function. Each component has actuarial interpretation and the model is analytic and continuous in mortality analysis. The analytic framework makes the function a unique keystone within the comprehensive class of mortality models particularly where a full age-range description is needed for life table graduation.

Heligman-Pollard (1980) proposed the eight-parameter mortality model assigned to describe the mortality odds of death as a continuous function of age. The function comprises three additive terms describing (1) early childhood mortality, (2) the accident hump in young adulthood and (3) adult mortality modelled as a Gompertz function. The parameters were estimated using weighted least-squares. The authors' work demonstrated some observable strengths capable of fitting mortality across the whole of the human age spectrum along a single functional form as opposed to some early models which focused only on some defined age bands. Each parameter is associated with some demographic phenomena in form of infant mortality decline and adult mortality hump. However, despite the strengths identified, the following limitations are observed. The entire work lacks analytic derivation from first principles and the analysis is purely empirical and full of demographic heuristics. It was not originally developed as a hazard function but it models death probabilities and hence this makes computation and some survival calculus intractable.

However, the original model suffers from high parameter inter-correlations, essentially when data are sparse or abridged, which hinders estimation in practice. Further analytic techniques have bolstered both the application and challenges of the Heligman-Pollard's function. A deep systematic review in Booth and Tickle (2008) stresses that the Heligman-Pollard's law is analytically distinguished among eight-parameter parametric laws by dealing with mortality trends from infancy to advance age within a single model. Its ability to fit different life tables is a proof of wide empirical application in demographic setting. However, this review also observes constraints (1) high correlations among parameters reduce forecasting robustness and in certain contexts, the model is less applicable in long-range projection compared to simple stochastic models. This implies that while Heligman-Pollard's law distinguishes itself analytically, its forecasting strength may be hampered unless parameters are constrained**.** Numerical **c**omparisons with other laws stress these limitations.

In a comparative study of parametric mortality models, the Siler model-a five-parameter combination of three basic hazard terms was observed to fit mortality data well or better than the eight-parameter Heligman-Pollard in certain populations with fewer parameters and less statistical instability**.** This result shows that Heligman-Pollard's flexibility may constitute a weakness as a result of potential misspecification or overparameterization, particularly when fitted to abridged life tables. Sharrow *et al.* (2013) employed advanced Bayesian computational strategy to the Heligman-Pollard function using rural South African data over the period of intense HIV mortality shifts. It computed parameter trajectories over time and compared changes to HIV prevalence. The Bayesian technique accounted for parameter uncertainty to easy out interpretation and forecasting reliability. By estimating parameters over time, the paper underscores how epidemic dynamics affect age-specific mortality trajectories. This is meant to demonstrate the use of the function beyond actuarial life tables, especially in high driven mortality contexts. However, the computational complexity of Bayesian technique needs careful prior numerical specification and are estimation-intensive hence this constitutes a barrier in actuarial analysis. The rural or incomplete death registration could bias the parameter estimation even in Bayesian framework.

Umar and Chukwudi (2019) applied Heligman-Pollard and Lee-Carter function to Nigerian mortality data, pointing out that though the Heligman-Pollard model seemed to overestimate old-age mortality, it nevertheless offers better overall forecast performance than Lee-Carter under their estimation technique. This comparison proves that Heligman-Pollard can generate good life tables or mortality forecast but its performance depends on data quality and estimation techniques particularly for populations with incomplete vital statistics.

Umar and Ugboh (2019) fits both Heligman-Pollard and Lee–Carter models to Nigerian mortality data and evaluates forecasting performance. The paper demonstrates some contextual relevance (1) the mortality model was applied in sub-saharan Africa where data limitations and mortality trend differ from high-income countries (2) the work shows that the model can outperform the Lee-Carter in some defined age groups and provided certain observable insight for actuarial scientists in data-poor environments. However, results show poor fit at older adult ages pointing to the structural weaknesses of the model in those ranges. Again, the incomplete vital registration in Nigeria would have introduced bias into the parameter estimation reducing its generalisation

Material and Methods

The Functional Nature of the Heligman–Pollard's Function

To enable us capture the underlying death pattern q_x from the crude death rates, the Heligman-Pollard class is defined as in equations (5). The Heligman-Pollard laws $\frac{q_x}{1-q_x}(HP_1)=\mu_x$, define continuous parsimonious functions of age such that the parameter vector adequately accounts for the trend of mortality to model human mortality throughout the entire lifespan. Each of the laws has three terms defining the three mortality periods in human life span: (1) infant mortality which rapidly decreases with age in the first few years of life (2) young adult mortality defining accident hump which rises as a result of accidents and (3) senescent mortality which exhibits exponential increase in mortality with age.

The third term of the second, third and fourth laws evolve because mortality decreases at highest ages. The parameters with demographic interpretation cannot be solved by any explicit method. Hence, the parameters can be estimated when the sum of squares

$$\ell(x)=\sum_{x=0}^{\Omega}\left(\frac{\bar{q}_x}{q_x}-1\right)^2 \tag{1}$$

is minimised using the Levenberg-Marquardt iteration techniques, where $\bar{q}_x$ is the death probability from Heligman-Pollard function and q_x is the observed death probability. In executing this method, mortality rate intensities, life annuity, life insurance functions and their trajectories were run on MATLAB.

Under the laws, the function μ_0 cannot assume age $x=0$ for a new born child because of the presence of $\log_e x$ and therefore to evaluate μ_0, a very small number $\varepsilon>0$ is chosen so that $\mu_0=\lim_{\varepsilon\to 0^+}\mu_\varepsilon$.

Method of Integrated Hazard

The integrated hazard function Λ_x is defined as

$$\Lambda_x=\int_0^x \mu_t dt \tag{2}$$

is the cumulative measure of mortality risk that a life is exposed to from birth to age x. In mortality analysis, Λ_x is applied to model the survival probability of the form

$$_t p_x=\exp(-\Lambda_x) \tag{3}$$

This exponential relationship which is generally analytically intractable for most parsimonious laws assumes a continuous-time survival process and forms the basis for many life table computations.

In the following theorems, the hazard rate function, the probability of survival function and survival function under each law was derived to distinguish the work from the previous studies. The integrated hazard function defines the total accumulated risk of death a life experiences from time 0 to t. Although the instantaneous mortality rate intensity μ_{x+t} can either increase or decrease due to trends, shocks or idiosyncratic risks, the integrated hazard function can only increase or remains constant. Since $\lim_{t\to\infty}({}_t p_x)=0$. Given sufficient time interval, the proportion of lives surviving diminishes to zero.

Survival Probability and The Heligman-Pollard's First Law

The Heligman-Pollard's laws each comprises three distinct components that collectively describe mortality patterns across ages. The Strab (2018) used some new modifications of the form $\log_e X^2$ to the second term of the original Heligman-Pollard's law in order to overcome the analytically complex integral. However, this study contributed fully based on existing literature by expanding the second term and then integrated it. The study did not originate the theorems on all the Heligman-Pollard functions but contributed and proved them based on the already established assumption stated in Bowers et al. (1997) that the survival probability is defined as

$$ {}_{\xi}p_x = \exp\left(-\int_0^{\xi}\mu_{x+s}ds\right) \qquad (4) $$

To the best of the author's knowledge, the survival probability theorems were not found in existing literatures for the Heligman-Pollard's first law.

Theorem

Given that the force of mortality at age x is

$$ \mu_x = A_1^{(x+A_2)^{A_3}} + B_1\exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) + C_1C_2^x \qquad (5) $$

then the survival probability function is given by

$$ {}_{t}p_x = \exp-\left\{\begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\dfrac{1}{A_3},(x+A_2)^{A_3}\log_e\dfrac{1}{A_1}\right)\\ -\Gamma\left(\dfrac{1}{A_3},(x+t+A_2)^{A_3}\log_e\dfrac{1}{A_1}\right)\end{array}\right\}+ \\ +\left[\dfrac{B_1(x+t)\mathrm{e}^{\left[-B_2\left[\log_e(x+t)\right]^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e(x+t)\right]}}{2B_2(\log_e B_3)-2B_2\left\{\log_e(x+t)\right\}}\right] \\ -\left[\dfrac{B_1 x\,\mathrm{e}^{\left(-B_2(\log_e x)^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e x\right)}}{2B_2(\log_e B_3)-2B_2\left\{\log_e x\right\}}\right] \\ +\left(C_1\dfrac{C_2^{x+t}}{\log_e C_2}-C_1\dfrac{C_2^{x}}{\log_e C_2}\right)\end{array}\right\} \qquad (6) $$

Following Heligman and Pollard (1980), A_1 is the level of infant mortality describing the probability of death q_1 between ages 1 and 2 years.

A_2 is the difference between death probabilities at age 0 and 1

A_3 describes the rate at which mortality declines throughout young ages

B_1 defines the intensity of accidental hump

B_2 represents the dispersion or spread of accidental hump

B_3 determines the location (measures of centrality) of accidental hump along the age x axis. A measure of this location is $mode_{hump} = \arg\max_x \mu_A(x) = B_3$, that is

$$ \frac{d}{dx}B_1\exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) = 0 \qquad (7) $$

$$ \Rightarrow x = B_3 \qquad (8) $$

C_1 describes the initial mortality at the beginning of senescence

C_2 quantifies the rate of ageing at senescent mortality.

C_3 governs the curvature of mortality curve at senescent ages.

In order to match the mortality odd $\frac{q_x}{1-q_x}$ with the Heligman-Pollard's mortality,

Debon et al (2005) defines $\frac{q_x}{1-q_x} = \mu_x$, called the mortality odd.

q_x is an increasing function since the probability of death increasing as age increases while the probability of survival p_x progressively decreases as age increases. Since $0 \le q_x \le 1$. It follows that $\frac{q_x}{p_x} = q_x + \varepsilon$, where $\varepsilon > 0$ is a very small real number and hence, it will not be too different from q_x. The essence of mortality odd is to restrict q_x from rising more than exponentially at the highest ages.

Proof

Expanding the first law in Heligman and Pollard (1980) at age $x+\zeta$, where $\zeta>0$, the mortality intensity is expressed as

$$\mu_{x+\zeta}=A_1^{\left(x+\zeta+A_2\right)^{A_3}}+B_1\,\mathrm{e}^{-B_2\left[\left(\log_e\left(x+\zeta\right)-\log_e B_3\right)^2\right]}+C_1C_2^{x+\zeta} \tag{9}$$

The auxiliary component mortality intensities can be expressed as

$$\mu_1\left(x+\zeta\right)=A_1^{\left(x+\zeta+A_2\right)^{A_3}} \tag{10}$$

$$\mu_2\left(x+\zeta\right)=B_1\,\mathrm{e}^{-B_2\left[\left(\log_e\left(x+\zeta\right)-\log_e B_3\right)^2\right]} \tag{11}$$

$$\mu_3\left(x+\zeta\right)=C_1C_2^{x+\zeta} \tag{12}$$

Following Strab (2018) and using equation (10), Define the transformation

$$\psi=\left(x+s+A_2\right)^{A_3} \tag{13}$$

$$\psi^{\frac{1}{A_3}}=\left(x+s+A_2\right) \tag{14}$$

Then, this implies

$$\psi^{-\frac{1}{A_3}}=\left(x+s+A_2\right)^{-1} \tag{15}$$

Differentiating ψ using (13) yields

$$\frac{d\psi}{ds}=A_3\left(x+s+A_2\right)^{A_3-1}=A_3\left(x+s+A_2\right)^{A_3}\left(x+s+A_2\right)^{-1} \tag{16}$$

$$\frac{d\psi}{ds}=A_3\frac{\psi}{\left(x+s+A_2\right)}=A_3\psi\frac{1}{\psi^{\frac{1}{A_3}}}=A_3\psi^{1-\frac{1}{A_3}}\Rightarrow d\psi=A_3\psi^{1-\frac{1}{A_3}}ds \tag{17}$$

$$\frac{d\psi}{A_3\psi^{1-\frac{1}{A_3}}}=ds \tag{18}$$

$$\mu_1\left(x+\zeta\right)=A_1^{\psi} \tag{19}$$

Integrate the first mortality component with respect to arbitrary time s to obtain

$$\Lambda_x=\int_0^t A_1^{\left(x+s+A_2\right)^{A_3}}ds=\int_{\left(x+A_2\right)^{A_3}}^{\left(x+t+A_2\right)^{A_3}}\left(\frac{A_1^{\psi}}{A_3\psi^{1-\frac{1}{A_3}}}\right)d\psi \tag{20}$$

$$\int_0^t A_1^{\left(x+s+A_2\right)^{A_3}}ds=\int_{\left(x+A_2\right)^{A_3}}^{\left(x+t+A_2\right)^{A_3}}\left(\frac{e^{\psi\log_e A_1}}{A_3\psi^{1-\frac{1}{A_3}}}\right)d\psi \tag{21}$$

$$\int_0^t A_1^{\left(x+s+A_2\right)^{A_3}}ds=\frac{1}{A_3}\int_{\left(x+A_2\right)^{A_3}}^{\left(x+t+A_2\right)^{A_3}}\left(\psi^{-1+\frac{1}{A_3}}e^{\psi\log_e A_1}\right)d\psi \tag{22}$$

Effect a second change of variable by letting,

$$\xi = \log_e\left(\frac{1}{A_1}\right)^{\psi} = \psi \log_e\left(\frac{1}{A_1}\right) = \psi \log_e\left(A_1\right)^{-1} = -\psi \log_e A_1 \tag{23}$$

$$\xi = -\psi \log_e A_1 \Rightarrow \xi = \log_e A_1^{-\psi} \Rightarrow e^{\xi} = A_1^{-\psi} \Rightarrow e^{-\xi} = A_1^{\psi} \tag{24}$$

$$\xi = -\psi \log_e A_1 \Rightarrow \frac{-\xi}{\log_e A_1} = \psi \tag{25}$$

$$\frac{d\xi}{d\psi} = -\log_e A_1 \Rightarrow \frac{d\xi}{-\log_e A_1} = d\psi \tag{26}$$

From equation (24), when

$$\psi = \left(x + A_2\right)^{A_3} \tag{27}$$

$$\xi = -\left(x + A_2\right)^{A_3}\left(\log_e A_1\right) \tag{28}$$

However, when

$$\psi = \left(x + t + A_2\right)^{A_3} \tag{29}$$

$$\xi = -\left(x + t + A_2\right)^{A_3}\left(\log_e A_1\right) \tag{30}$$

Therefore, equation (22) transforms to (10)

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \int_{(x+A_2)^{A_3}}^{(x+t+A_2)^{A_3}} \left(\frac{A_1^{\psi}}{A_3 \psi^{1-\frac{1}{A_3}}}\right) d\psi \tag{31}$$

$$= \frac{-1}{A_3}\int_{-(x+A_2)^{A_3}\log_e A_1}^{-(x+t+A_2)^{A_3}\log_e A_1} \left(\left(\frac{-\xi}{\log_e A_1}\right)^{-1+\frac{1}{A_3}} e^{-\xi}\right)\frac{d\xi}{\log_e A_1}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{-1}{A_3} \times \frac{1}{\log_e A_1}\int_{-(x+A_2)^{A_3}\log_e A_1}^{-(x+t+A_2)^{A_3}\log_e A_1} \left(\left(\frac{-1}{\log_e A_1}\right)^{-1+\frac{1}{A_3}} (\xi)^{-1+\frac{1}{A_3}} e^{-\xi}\right) d\xi \tag{32}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{-1}{A_3} \times \left(\frac{1}{-\log_e A_1}\right)^{-1} \times \left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \times \frac{1}{\log_e A_1}\int_{-(x+A_2)^{A_3}\log_e A_1}^{-(x+t+A_2)^{A_3}\log_e A_1} \left((\xi)^{-1+\frac{1}{A_3}} e^{-\xi}\right) d\xi \tag{33}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{-1}{A_3} \times \left(-\log_e A_1\right) \times \left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \times \frac{1}{\log_e A_1}\int_{-(x+A_2)^{A_3}\log_e A_1}^{-(x+t+A_2)^{A_3}\log_e A_1} \left((\xi)^{-1+\frac{1}{A_3}} e^{-\xi}\right) d\xi \tag{34}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \times \left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \times \int_{-(x+A_2)^{A_3}\log_e A_1}^{-(x+t+A_2)^{A_3}\log_e A_1} \left((\xi)^{\frac{1}{A_3}-1} e^{-\xi}\right) d\xi \tag{35}$$

Following definitions in Watson (2015) and Masina (2019)

$\gamma(z,x) = \int\limits_0^x t^{z-1} e^{-t} dt$ is the incomplete lower Gamma integral

$\Gamma(z,x) = \int\limits_x^{\infty} t^{z-1} e^{-t} dt$ is the incomplete upper Gamma integral

$\Gamma(z) = \int\limits_0^{\infty} t^{z-1} e^{-t} dt$ is the Gamma integral

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left((\xi)^{\frac{1}{A_3}-1} e^{-\xi} \right) d\xi \tag{36}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \int_{(-1)(x+A_2)^{A_3} \log_e A_1}^{\infty} \left(e^{-\xi} (\xi)^{\frac{1}{A_3}-1} \right) d\xi \\ -\int_{(-1)(x+t+A_2)^{A_3} \log_e A_1}^{\infty} \left(e^{-\xi} (\xi)^{\frac{1}{A_3}-1} \right) d\xi \end{array} \right\} \tag{37}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \\ -\Gamma\left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \end{array} \right\} \tag{38}$$

Let $z = \frac{1}{A_3} > 0$ and

$$y = (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \tag{39}$$

But in mortality, the gamma function can be defined as

$$G(z,y) = \frac{1}{\Gamma(z)} \int_0^y t^{z-1} e^{-t} dt \tag{40}$$

Consequently,

$$\Gamma(z,y) = \Gamma(z) - \Gamma(z) \times G(z,y) \tag{41}$$

Integrating the second term teenage mortality component with respect to arbitrary time s,

$$\mu_2(x+s) = B_1 \mathrm{e}^{-B_2\left[\left(\log_e(x+s) - \log_e B_3 \right)^2 \right]} \tag{42}$$

Expanding the exponent in equation (42) yields

$$\mu_2(x+s) = B_1 \mathrm{e}^{\left[-B_2\left[\log_e(x+s) \right]^2 - B_2\left[\log_e B_3 \right]^2 + 2B_2(\log_e B_3)\log_e(x+s) \right]} \tag{43}$$

The auxiliary hazard function yields equation (44),

$$\int_0^t \mu_2(x+s) ds = \left[\frac{B_1 \mathrm{e}^{\left[-B_2\left[\log_e(x+s) \right]^2 - B_2\left[\log_e B_3 \right]^2 + 2B_2(\log_e B_3)\log_e(x+s) \right]}}{-2B_2 \left\{ \log_e(x+s) \right\} \times \frac{1}{(x+s)} + 2B_2(\log_e B_3) \times \frac{1}{(x+s)}} \right]_0^t \tag{44}$$

$$\int_0^t \mu_2(x+s) ds = \left[\frac{B_1 \mathrm{e}^{\left[-B_2\left[\log_e(x+s) \right]^2 - B_2\left[\log_e B_3 \right]^2 + 2B_2(\log_e B_3)\log_e(x+s) \right]}}{2B_2(\log_e B_3) - 2B_2 \left\{ \log_e(x+s) \right\}} \times (x+s) \right]_0^t \tag{45}$$

$$\int_0^t \mu_2(x+s) ds = \left[\frac{B_1 (x+t) \mathrm{e}^{\left[-B_2\left[\log_e(x+t) \right]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e(x+t) \right]}}{2B_2(\log_e B_3) - 2B_2 \left\{ \log_e(x+t) \right\}} \right] - \left[\frac{B_1 x \mathrm{e}^{\left(-B_2(\log_e x)^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e x \right)}}{2B_2(\log_e B_3) - 2B_2 \left\{ \log_e x \right\}} \right] \tag{46}$$

Integrate the third component with respect to s yields

$$C_1\int_0^t C_2^{x+s}ds = C_1\left.\frac{C_2^{x+s}}{\log_e C_2}\right|_0^t = C_1\frac{C_2^{x+t}}{\log_e C_2} - C_1\frac{C_2^{x}}{\log_e C_2} \tag{47}$$

$$\int_0^t \mu_3(x+s)ds = \left(C_1\frac{C_2^{x+t}}{\log_e C_2} - C_1\frac{C_2^{x}}{\log_e C_2}\right) \tag{48}$$

The total severity to die function within $0 \le s \le t$ is therefore

$$\int_0^t \mu(x+s)ds = \int_0^t \left\{A_1^{(x+s+A_2)^{A_3}} + B_1\exp\left(-B_2\log_e\left(\frac{x+s}{B_3}\right)^2\right) + C_1C_2^{x+s}\right\}ds \tag{49}$$

$$\int_0^t \mu_{x+s}ds = \left\{\begin{array}{l} \frac{1}{A_3}\times\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\frac{1}{A_3},(x+A_2)^{A_3}\log_e\frac{1}{A_1}\right)\\ -\Gamma\left(\frac{1}{A_3},(x+t+A_2)^{A_3}\log_e\frac{1}{A_1}\right)\end{array}\right\}+ \\ \left[\frac{B_1(x+t)\mathrm{e}^{\left[-B_2\left[\log_e(x+t)\right]^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e(x+t)\right]}}{2B_2(\log_e B_3)-2B_2\{\log_e(x+t)\}}\right] \\ -\left[\frac{B_1x\,\mathrm{e}^{\left(-B_2(\log_e x)^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e x\right)}}{2B_2(\log_e B_3)-2B_2\{\log_e x\}}\right] \\ +\left(C_1\frac{C_2^{x+t}}{\log_e C_2}-C_1\frac{C_2^{x}}{\log_e C_2}\right)\end{array}\right\} \tag{50}$$

the probability that a life age x survives to the next age $(x+t)$ is obtained as

$${}_t p_x = \exp-\left\{\begin{array}{l} \frac{1}{A_3}\times\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\frac{1}{A_3},(x+A_2)^{A_3}\log_e\frac{1}{A_1}\right)\\ -\Gamma\left(\frac{1}{A_3},(x+t+A_2)^{A_3}\log_e\frac{1}{A_1}\right)\end{array}\right\}+ \\ \left[\frac{B_1(x+t)\mathrm{e}^{\left[-B_2\left[\log_e(x+t)\right]^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e(x+t)\right]}}{2B_2(\log_e B_3)-2B_2\{\log_e(x+t)\}}\right] \\ -\left[\frac{B_1x\,\mathrm{e}^{\left(-B_2(\log_e x)^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e x\right)}}{2B_2(\log_e B_3)-2B_2\{\log_e x\}}\right] \\ +\left(C_1\frac{C_2^{x+t}}{\log_e C_2}-C_1\frac{C_2^{x}}{\log_e C_2}\right)\end{array}\right\} \tag{51}$$

From the arguments in Gavrilova *et al.* (2017), Gavrilov and Gavrilova (2019a), Gavrilov; Gavrilova (2019b) and Dang *et al.* (2023), all Heligman-Pollard's laws have increasing hazard function.

This completes the proof. The proposition below is elegantly stated to unify all the laws of mortality.

Proposition: **The Universal Law of Mortality** A life after birth continues to be in a state of vitality $\nu(t)$ or of continuous longevity unless otherwise acted against his survival by an external force of mortality μ_x due to a defined cause τ. This law implies that (x) dies when $\nu(t)=0$

Investigating the Limiting Properties of the Survival Probability Function

The goal of this section is to confirm the asymptotic properties $\lim\limits_{\xi\to\infty}\left({}_{\xi}p_x\right)=0$ and $\lim\limits_{\xi\to 0}\left({}_{\xi}p_x\right)=1$ of the survival probability function stated in Bowers et al. (1997) under each Heligman-Pollard's law. Taking the limit in equation (51) as time approaches zero yields

$$\lim_{t\to 0}\left({}_{t}p_x\right)=\lim_{t\to 0}\exp-\left\{\begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\dfrac{1}{A_3},\left(x+A_2\right)^{A_3}\log_e\dfrac{1}{A_1}\right)\\ -\Gamma\left(\dfrac{1}{A_3},\left(x+A_2\right)^{A_3}\log_e\dfrac{1}{A_1}\right)\end{array}\right\}\\ +\left[\dfrac{B_1 x\,\mathrm{e}^{\left[-B_2\left[\log_e(x)\right]^2-B_2\left(\log_e B_3\right)^2+2B_2\left(\log_e B_3\right)\log_e x\right]}}{2B_2\left(\log_e B_3\right)-2B_2\left\{\log_e x\right\}}\right]\\ -\left[\dfrac{B_1 x\,\mathrm{e}^{\left(-B_2\left(\log_e x\right)^2-B_2\left(\log_e B_3\right)^2+2B_2\left(\log_e B_3\right)\log_e x\right)}}{2B_2\left(\log_e B_3\right)-2B_2\left\{\log_e x\right\}}\right]\\ +\left(C_1\dfrac{C_2^{x}}{\log_e C_2}-C_1\dfrac{C_2^{x}}{\log_e C_2}\right)\end{array}\right\}=1 \qquad (52)$$

Since all the terms inside the braces sum to zero in equation (52) as the time of survival approaches $t=0$, the probability that a new born will survive is 1, everything else being equal. Furthermore, the limit as time approaches infinity yields equation (53).

$$\lim_{t\to \infty}\left({}_{t}p_x\right)=\lim_{t\to \infty}\exp-\left\{\begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\dfrac{1}{A_3},\left(x+A_2\right)^{A_3}\log_e\dfrac{1}{A_1}\right)\\ -\Gamma\left(\dfrac{1}{A_3},\left(x+t+A_2\right)^{A_3}\log_e\dfrac{1}{A_1}\right)\end{array}\right\}\\ +\left[\dfrac{B_1\left(x+t\right)\mathrm{e}^{\left[-B_2\left[\log_e(x+t)\right]^2-B_2\left(\log_e B_3\right)^2+2B_2\left(\log_e B_3\right)\log_e(x+t)\right]}}{2B_2\left(\log_e B_3\right)-2B_2\left\{\log_e\left(x+t\right)\right\}}\right]\\ -\left[\dfrac{B_1 x\,\mathrm{e}^{\left(-B_2\left(\log_e x\right)^2-B_2\left(\log_e B_3\right)^2+2B_2\left(\log_e B_3\right)\log_e x\right)}}{2B_2\left(\log_e B_3\right)-2B_2\left\{\log_e x\right\}}\right]\\ +\left(C_1\dfrac{C_2^{x+t}}{\log_e C_2}-C_1\dfrac{C_2^{x}}{\log_e C_2}\right)\end{array}\right\}=e^{-\infty}=0 \qquad (53)$$

Since all the terms in the braces sum to infinity in equation (53), the probability that a life aged x will survive to the highest age is 0, everything else being equal. Using the consequences of the mortality odd implies that

$$\frac{p_x}{1-p_x} = A_1^{(x+A_2)^{A_3}} + B_1 \exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) + C_1 C_2^x \tag{54}$$

Then the probability that a life aged x survives to the next age $x+1$ becomes

$$p_x = \frac{A_1^{(x+A_2)^{A_3}} + B_1 \exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) + C_1 C_2^x}{1 + A_1^{(x+A_2)^{A_3}} + B_1 \exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) + C_1 C_2^x} \tag{55}$$

Theorem

The goal here is to obtain the survival function from the corresponding probability of survival under the first law.
Given that the force of mortality at age y,

$$\mu_x = \frac{q_x}{1-q_x} = A_1^{(x+A_2)^{A_3}} + B_1 \exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) + C_1 C_2^x \tag{56}$$

then, the survival function representing the number of lives at age y

$$\int_0^\infty l_{y+t}\mu_{y+t}\,dt = l_0 \exp - \left\{ \begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\dfrac{1}{A_3}, (A_2)^{A_3} \log_e \dfrac{1}{A_1}\right) \\ -\Gamma\left(\dfrac{1}{A_3}, (y+A_2)^{A_3} \log_e \dfrac{1}{A_1}\right) \end{array} \right\} \\ + \left[\dfrac{B_1 \times y \times e^{\left[-B_2[\log_e y]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e y\right]}}{2B_2\left(\log_e B_3\right) - 2B_2\left\{\log_e y\right\}} \right] \\ + \left(C_1 \dfrac{C_2^t}{\log_e C_2} - C_1 \dfrac{1}{\log_e C_2} \right) \end{array} \right\} \tag{57}$$

Proof
Recall that

$$
{}_t p_x = \exp - \left\{ \begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\dfrac{1}{A_3},\left(x+A_2\right)^{A_3}\log_e \dfrac{1}{A_1}\right) \\ -\Gamma\left(\dfrac{1}{A_3},\left(x+t+A_2\right)^{A_3}\log_e \dfrac{1}{A_1}\right) \end{array} \right\} \\ +\left[\dfrac{B_1\left(x+t\right)\mathrm{e}^{\left[-B_2\left[\log_e\left(x+t\right)\right]^2 - B_2\left(\log_e B_3\right)^2 + 2B_2\left(\log_e B_3\right)\log_e\left(x+t\right)\right]}}{2B_2\left(\log_e B_3\right) - 2B_2\left\{\log_e\left(x+t\right)\right\}} \right] \\ -\left[\dfrac{B_1 x\,\mathrm{e}^{\left(-B_2\left(\log_e x\right)^2 - B_2\left(\log_e B_3\right)^2 + 2B_2\left(\log_e B_3\right)\log_e x\right)}}{2B_2\left(\log_e B_3\right) - 2B_2\left\{\log_e x\right\}} \right] \\ +\left(C_1 \dfrac{C_2^{x+t}}{\log_e C_2} - C_1 \dfrac{C_2^{x}}{\log_e C_2} \right) \end{array} \right\} \quad (58)
$$

Equation (58) yields equation (59)

$$
{}_t p_0 = \exp - \left\{ \begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\dfrac{1}{A_3},\left(A_2\right)^{A_3}\log_e \dfrac{1}{A_1}\right) \\ -\Gamma\left(\dfrac{1}{A_3},\left(t+A_2\right)^{A_3}\log_e \dfrac{1}{A_1}\right) \end{array} \right\} \\ +\left[\dfrac{B_1\times t\times\mathrm{e}^{\left[-B_2\left[\log_e t\right]^2 - B_2\left(\log_e B_3\right)^2 + 2B_2\left(\log_e B_3\right)\log_e t\right]}}{2B_2\left(\log_e B_3\right) - 2B_2\left\{\log_e t\right\}} \right] \\ +\left(C_1 \dfrac{C_2^{t}}{\log_e C_2} - C_1 \dfrac{1}{\log_e C_2} \right) \end{array} \right\} \quad (59)
$$

Therefore, the survival probability yields

$$
l_t = l_0 \ \exp - \left\{ \begin{array}{l} \dfrac{1}{A_3}\left(\dfrac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\dfrac{1}{A_3},\left(A_2\right)^{A_3}\log_e \dfrac{1}{A_1}\right) \\ -\Gamma\left(\dfrac{1}{A_3},\left(t+A_2\right)^{A_3}\log_e \dfrac{1}{A_1}\right) \end{array} \right\} \\ +\left[\dfrac{B_1 t\,\mathrm{e}^{\left[-B_2\left[\log_e t\right]^2 - B_2\left(\log_e B_3\right)^2 + 2B_2\left(\log_e B_3\right)\log_e t\right]}}{2B_2\left(\log_e B_3\right) - 2B_2\left\{\log_e t\right\}} \right] \\ +\left(C_1 \dfrac{C_2^{t}}{\log_e C_2} - C_1 \dfrac{1}{\log_e C_2} \right) \end{array} \right\} \quad (60)
$$

Since t is arbitrary, it follows that t can be replaced by y to obtain the number of lives expected to survive at age y. By definition, the survival function is given by $\int_0^\infty l_{y+t}\mu_{y+t}dt = l_y$

$$l_y = l_0 \exp - \left\{ \begin{array}{l} \frac{1}{A_3}\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\frac{1}{A_3}, (A_2)^{A_3} \log_e \frac{1}{A_1}\right) \\ -\Gamma\left(\frac{1}{A_3}, (y+A_2)^{A_3} \log_e \frac{1}{A_1}\right) \end{array} \right\} \\ + \left[\frac{B_1 y\, e^{\left[-B_2[\log_e y]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e y\right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e y\}} \right] \\ + \left(C_1 \frac{C_2^y}{\log_e C_2} - C_1 \frac{1}{\log_e C_2} \right) \end{array} \right\} \tag{61}$$

This completes the proof. ${}_n d_x = l_x - l_{x+n}$

$$\begin{array}{l} d_y = l_0 \exp - \left\{ \begin{array}{l} \frac{1}{A_3}\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\frac{1}{A_3}, (A_2)^{A_3} \log_e \frac{1}{A_1}\right) \\ -\Gamma\left(\frac{1}{A_3}, (y+A_2)^{A_3} \log_e \frac{1}{A_1}\right) \end{array} \right\} \\ + \left[\frac{B_1 \times y \times e^{\left[-B_2[\log_e y]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e y\right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e y\}} \right] \\ + \left(C_1 \frac{C_2^t}{\log_e C_2} - C_1 \frac{1}{\log_e C_2} \right) \end{array} \right\} \\ -l_0 \exp - \left\{ \begin{array}{l} \frac{1}{A_3}\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}} \left\{ \begin{array}{l} \Gamma\left(\frac{1}{A_3}, (A_2)^{A_3} \log_e \frac{1}{A_1}\right) \\ -\Gamma\left(\frac{1}{A_3}, ((y+n)+A_2)^{A_3} \log_e \frac{1}{A_1}\right) \end{array} \right\} \\ + \left[\frac{B_1 \times (y+n) \times e^{\left[-B_2[\log_e(y+n)]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e(y+n)\right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e (y+n)\}} \right] \\ + \left(C_1 \frac{C_2^t}{\log_e C_2} - C_1 \frac{1}{\log_e C_2} \right) \end{array} \right\} \end{array} \tag{62}$$

$$
{}_{n}q_{y}=\frac{l_0\exp-\left\{\begin{array}{l}\frac{1}{A_3}\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\frac{1}{A_3},\left(A_2\right)^{A_3}\log_e\frac{1}{A_1}\right)\\-\Gamma\left(\frac{1}{A_3},\left(y+A_2\right)^{A_3}\log_e\frac{1}{A_1}\right)\end{array}\right\}\\+\left[\frac{B_1\times y\times \mathrm{e}^{\left[-B_2[\log_e y]^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e y\right]}}{2B_2\left(\log_e B_3\right)-2B_2\left\{\log_e y\right\}}\right]\\+\left(C_1\frac{C_2^t}{\log_e C_2}-C_1\frac{1}{\log_e C_2}\right)\end{array}\right\}-l_0\exp-\left\{\begin{array}{l}\frac{1}{A_3}\left(\frac{1}{-\log_e A_1}\right)^{\frac{1}{A_3}}\left\{\begin{array}{l}\Gamma\left(\frac{1}{A_3},\left(A_2\right)^{A_3}\log_e\frac{1}{A_1}\right)\\-\Gamma\left(\frac{1}{A_3},\left(\left(y+n\right)+A_2\right)^{A_3}\log_e\frac{1}{A_1}\right)\end{array}\right\}\\+\left[\frac{B_1\times\left(y+n\right)\times \mathrm{e}^{\left[-B_2\left[\log_e(y+n)\right]^2-B_2(\log_e B_3)^2+2B_2(\log_e B_3)\log_e(y+n)\right]}}{2B_2\left(\log_e B_3\right)-2B_2\left\{\log_e\left(y+n\right)\right\}}\right]\\+\left(C_1\frac{C_2^t}{\log_e C_2}-C_1\frac{1}{\log_e C_2}\right)\end{array}\right\}}{l_y} \quad (63)
$$

Presentation of Results

The special mathematical functions applied have the following consequences on the death severity function. Using equation (5), the integrated hazard function is

$$\Lambda_x=\int_{x_0}^{x}\mu_t dt \quad (64)$$

The function is decomposed into three auxiliary components $\Lambda_i\left(x\right);i=1,2,3$

$$\Lambda_1\left(x\right)=\int_{x_0}^{x}A_1^{\left(t+A_2\right)^{A_3}}dt \quad (65)$$

The integrand is analytic in t. Since $A_1>0$,

$$A_1^{\left(t+A_2\right)^{A_3}}=\exp\left(t+A_2\right)^{A_3}\ln A_1 \quad (66)$$

This function is smooth and analytic on $\left(0,\infty\right)$, hence $\Lambda_1\left(x\right)$ is analytic for $x>0$. An immediate consequence here is that $\left(t+A_2\right)^{A_3}$ can be estimated using the Taylor's series provided $A_3\in\mathbb{Q}$ allowing analytic continuation in the complex domain for mortality evaluation under fractional exponents.

$$\Lambda_2(x) = \int_{x_0}^{x} B_1 \exp\left[-B_2(\ln t - \ln B_3)^2\right] dt \quad (67)$$

Setting $v = \ln t$ yields

$$\Lambda_2(x) = \int_{\ln x_0}^{\ln x} B_1 \exp\left[-B_2(\ln t - \ln B_3)^2\right] e^v dv \quad (68)$$

This is a weighted Gaussian integral. However, the term $e^{\left[-B_2(\ln t - \ln B_3)^2\right]} e^v$ is analytic in v and thus in x, hence $\Lambda_2(x)$ is smooth, analytic and can be bounded for any finite age x. The function $\Lambda_2(x)$ is clearly dominated by a lognormal term implying that its hazard rate function peaks at $x = B_3$ but diminishes symmetrically in log-space and hence permits innovative estimations using log-normal moments.

$$\Lambda_3(x) = \int_{x_0}^{x} C_1 C_x^t dt \quad (69)$$

Since $C_2^{\,t} = \exp(t \ln C_2)$, hence

$$\Lambda_3(x) = \frac{C_1}{\ln C_2}\left(C^x - C^{x_0}\right) \quad (70)$$

This implies that $\Lambda_3(x)$ grows exponentially with age and its derivative agrees with Gompertz's law. This can be applied to compare the tail behaviour of different parametric mortality laws.

Table 1: Mortality Table

x	$HP1(Male)$	$HP1(Female)$
0	0.145820	0.219396
1	0.007861	0.007446
2	0.002271	0.001811
3	0.000978	0.000688
4	0.000536	0.000347
5	0.000355	0.000221
6	0.000274	0.000171
7	0.000238	0.000151
8	0.000224	0.000146
9	0.000224	0.000149
10	0.000233	0.000156
11	0.000251	0.000166
12	0.000281	0.000178
13	0.000326	0.000192
14	0.000393	0.000209
15	0.000482	0.000231
16	0.000593	0.000259
17	0.000724	0.000296
18	0.000868	0.000345
19	0.001018	0.000409
20	0.001165	0.000489
21	0.001302	0.000582
22	0.001424	0.000686
23	0.001527	0.000797
24	0.001611	0.000907
25	0.001677	0.001014
26	0.001728	0.001111
27	0.001767	0.001197

28	0.001798	0.001269
29	0.001825	0.001329
30	0.001853	0.001378
31	0.001886	0.001419
32	0.001926	0.001455
33	0.001978	0.001489
34	0.002042	0.001526
35	0.002122	0.001568
36	0.002219	0.001618
37	0.002333	0.001678
38	0.002468	0.001750
39	0.002623	0.001837
40	0.002800	0.001938
41	0.003000	0.002055
42	0.003224	0.002190
43	0.003474	0.002342
44	0.003751	0.002513
45	0.004056	0.002704
46	0.004393	0.002915
47	0.004762	0.003148
48	0.005166	0.003405
49	0.005609	0.003686
50	0.006092	0.003993
51	0.006619	0.004328
52	0.007195	0.004694
53	0.007821	0.005092
54	0.008504	0.005525
55	0.009248	0.005996
56	0.010058	0.006508
57	0.010940	0.007064
58	0.011899	0.007668
59	0.012943	0.008325
60	0.014080	0.009037
61	0.015316	0.009812
62	0.016661	0.010652
63	0.018125	0.011565
64	0.019717	0.012557
65	0.021450	0.013633
66	0.023334	0.014801
67	0.025385	0.016070
68	0.027616	0.017448
69	0.030043	0.018944
70	0.032683	0.020568
71	0.035555	0.022331
72	0.038680	0.024246
73	0.042080	0.026324
74	0.045778	0.028581
75	0.049801	0.031031
76	0.054178	0.033692
77	0.058940	0.036580
78	0.064120	0.039717

79	0.069756	0.043122
80	0.075887	0.046819
81	0.082556	0.050832
82	0.089812	0.055190
83	0.097706	0.059922
84	0.106293	0.065059
85	0.115635	0.070637
86	0.125799	0.076693
87	0.136855	0.083268
88	0.148883	0.090407
89	0.161968	0.098158
90	0.176204	0.106574
91	0.191690	0.115711
92	0.208538	0.125631
93	0.226866	0.136402
94	0.246806	0.148096
95	0.268498	0.160792
96	0.292096	0.174578
97	0.317768	0.189545
98	0.345697	0.205795
99	0.376080	0.223439
100	0.409134	0.242595
101	0.445092	0.263393
102	0.484212	0.285975
103	0.526769	0.310493
104	0.573067	0.337112
105	0.623434	0.366014
106	0.678227	0.397394
107	0.737836	0.431463
108	0.802685	0.468454
109	0.873233	0.508616
110	0.949981	0.552222
111	1.033475	0.599566
112	1.124307	0.650968
113	1.223123	0.706778
114	1.330623	0.767373
115	1.447571	0.833162
116	1.574798	0.904592
117	1.713208	0.982146
118	1.863781	1.066348
119	2.027589	1.157770
120	2.205794	1.257030

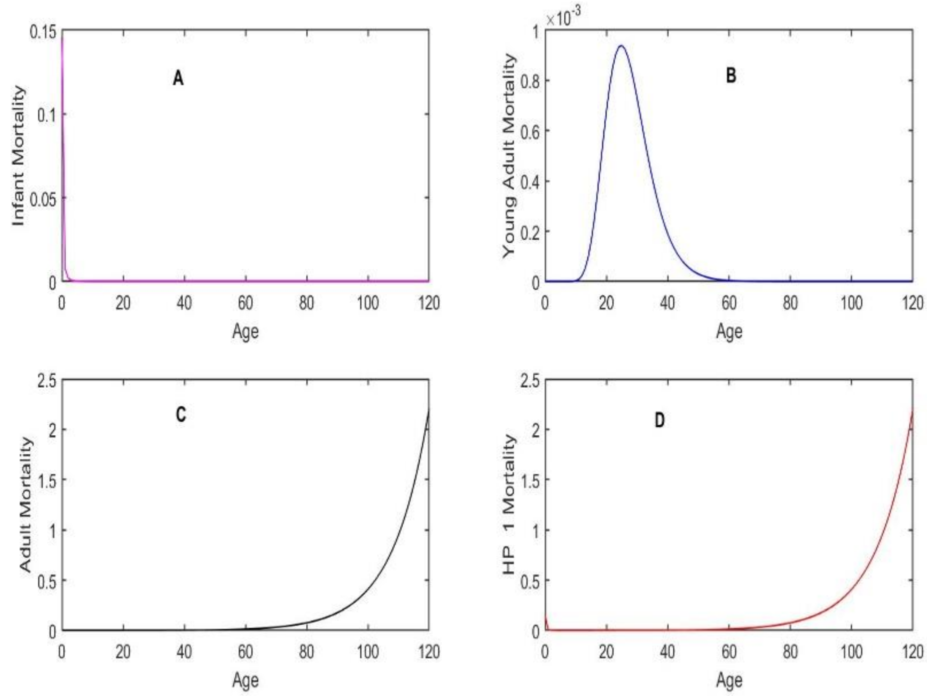

Figure 1 HP 1 Male Mortality Component Curves for Infant (A), Young Adult (B), Adult (C) and HP1 (D)

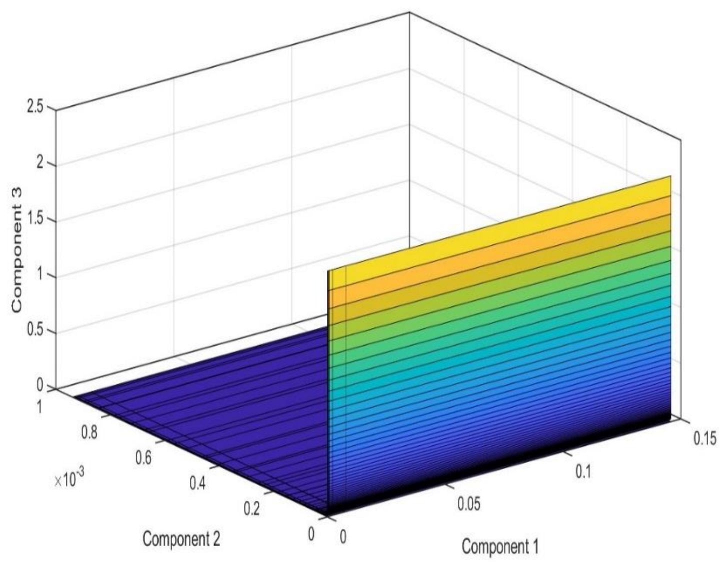

Figure 2 HP 1 Male Mortality Surface

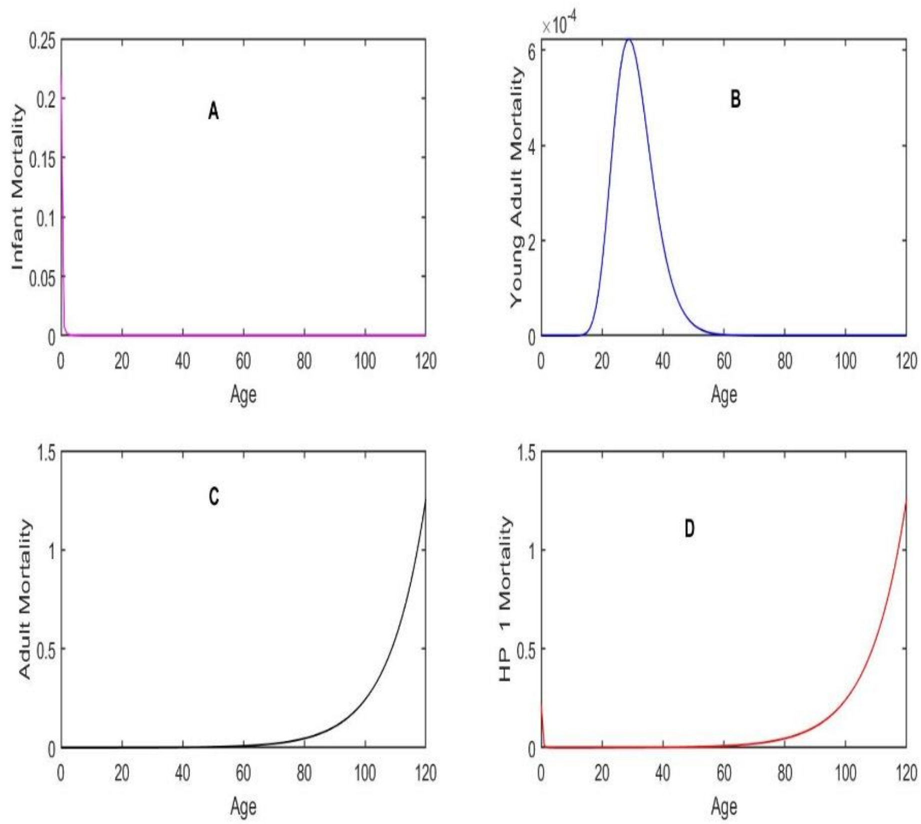

Figure 3 HP 1 Female Mortality Mortality Component Curves for Infant (A),Young Adult (B), Adult (C) and HP1 (D)

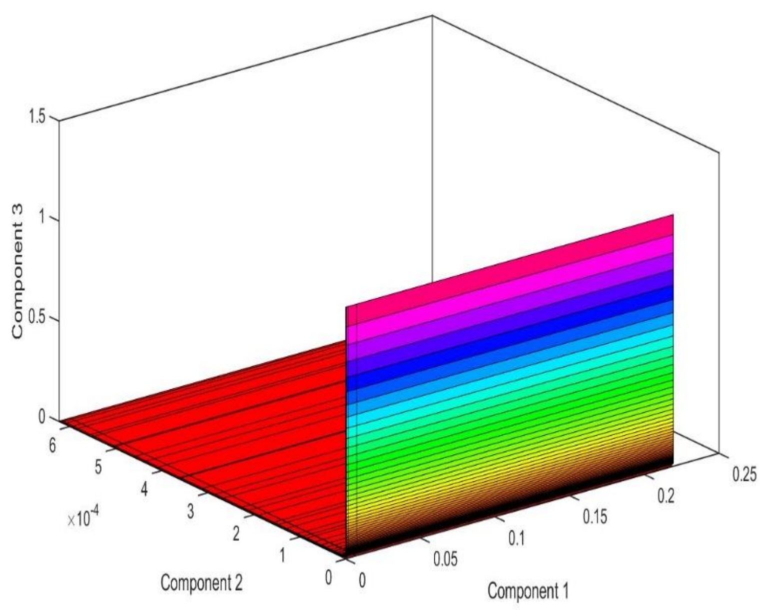

Figure 4 HP 1 Female Mortality Surface

Discussion of Results

The discussions given under are both analytic and empirical. The gamma function and Euler-Maclaurin series expansion were applied to obtain the Heligman-Pollard hazard rate functions derived in equations (50)-(63). The Euler-Maclaurin series expansion is a well-grounded method of approximating sums by integrals with quantifiable error bounds and lends analytic robustness to the expressions. The Euler-Maclaurin series provides known error bounds under certain smoothness conditions of the mortality laws and interest functions. Consequently, the models maintain a high level of analytic accuracy, provided that assumptions on smoothness hold. The symbolic and numerical computations under constructed Heligman-Pollard's laws proved that the new models behaved consistently with established theoretical expectations that actuarial hazard rate function increased with age. Based on the analytic special functions applied, the survival probability functions were derived. Each auxiliary hazard term $\Lambda_x \in C^{\infty}\left(0,\infty\right)$. Again, $\mu_x > 0$, $\forall x > 0$, then $\Lambda'_x = \mu_x > 0$, this implies Λ_x is strictly increasing. Again, $\Lambda''_x = \mu'_x$, however, at very young ages, the derivative of infant mortality term could decline possibly showing non-convexity at infancy. For large, x, $\mu_x \to C_1 C_2^x$ and consequently, $\Lambda_x \to \dfrac{C_1}{\ln C_2} C_2^x$, $x \to \infty$, this implies $\Lambda_x \to \infty$ exponentially. However, based on analytic properties identified, the following function can be used as a possible approximation.

The survival probabilities under the Heligman-Pollard's laws were proved to satisfy the asymptotic properties at age zero of a new born and at highest mortality age in equations (52) and (53) respectively, however, its mortality rate intensities cannot directly admit the age of new born at zero, making it asymptotically unstable. The mortality rate intensity at perinatality in all the Heligman-Pollard's laws was inadmissible because as the age $x \to 0^+$ in the second terms of the four laws hence, $\lim\limits_{x \to 0^+} \log_e x \to -\infty$. As a result of this, the mortality slope a μ'_x did not possess global existence. In order to overcome this problem not observed in Emilidha and Danardono (2017) and Sari, Deautama and Febrisutisyanto (2023), a small number $\varepsilon = 0.00001$ was chosen to replace $x = 0$ to enable a rough estimation of the perinatal mortality rates when preparing mortality tables. The Heligman-Pollard's laws have parameters that are awfully too low or too high.

Recall that

$$\mu_x = \frac{q_x}{1-q_x} = A_1^{(x+A_2)^{A_3}} + B_1 \exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) + C_1 C_2^x \in \left(0,\infty\right) \qquad (71)$$

Define the first component in Figures 1 and 3

$$\frac{q_x^{(1)}}{1-q_x^{(1)}} = A_1^{(x+A_2)^{A_3}} \qquad (72)$$

Define the second component in Figures 1 and 3

$$\frac{q_x^{(2)}}{1-q_x^{(2)}} = B_1 \exp\left(-B_2\left(\log_e x - \log_e B_3\right)^2\right) \qquad (73)$$

Define the third component in Figure 1 and 3

$$\frac{q_x^{(3)}}{1-q_x^{(3)}} = C_1 C_2^x \qquad (74)$$

The mortality is non-negative, finite for relevant ages, dimentionless and summable.

Equation (70) satisfies that $x+A_2>0$. Consequently, $x\geq 0$ implies that $A_2>0$. Hence $\left(x+A_2\right)^{A_3}\in\left(0,\infty\right)>0$ and $\frac{q_x}{1-q_x}>0$

$$\ln\frac{q_x^{(1)}}{1-q_x^{(1)}}=\ln A_1^{\left(x+A_2\right)^{A_3}} \tag{75}$$

$$\ln\frac{q_x^{(1)}}{1-q_x^{(1)}}=\left(\left(x+A_2\right)^{A_3}\right)\ln A_1 \tag{76}$$

$$\lim_{x\to 0}\left(\ln\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)=\lim_{x\to 0}\left(\left(x+A_2\right)^{A_3}\right)\ln A_1 \tag{77}$$

$$\lim_{x\to 0}\left(\ln\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)=\left(A_2\right)^{A_3}\ln A_1=\ln\left(A_1\right)^{\left(A_2\right)^{A_3}} \tag{78}$$

Therefore, from the first component in Figure 1, the mortality rate at birth coincides with the vertical axis. However, when the insured advances in age to infinity, we have that

$$\lim_{x\to\infty}\left(x+A_2\right)^{A_3}=\infty \tag{79}$$

and since $0<A_1<1$

$$\lim_{x\to\infty}\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)=\left(A_1\right)^{\infty}=0 \tag{80}$$

$$\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)=0;\quad x\to\infty \tag{81}$$

From same Figure 1, the mortality rate at birth declines and falls to the age axis. Now, the mortality slope is given as

$$\frac{d}{dx}\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)=A_3\left(x+A_2\right)^{A_3-1}A_1^{\left(x+A_2\right)^{A_3}}\ln A_1 \tag{82}$$

$A_3>0$; $\ln A_1<0$ and $\left(x+A_2\right)^{A_3-1}>0$

Therefore,

$\frac{d}{dx}\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)<0;\quad\forall x$ and $\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)$ is decreasing. The mortality curvature is

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)= \tag{83}$$
$$\left(\ln A_1\right)A_3\left(x+A_2\right)^{A_3-1}\frac{d}{dx}A_1^{\left(x+A_2\right)^{A_3}}+\left(\ln A_1\right)A_3A_1^{\left(x+A_2\right)^{A_3}}\frac{d}{dx}\left(x+A_2\right)^{A_3-1}$$

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)=$$
$$\left(\ln A_1\right)A_3\left(x+A_2\right)^{A_3-1}\left[A_3\left(x+A_2\right)^{A_3-1}A_1^{\left(x+A_2\right)^{A_3}}\ln A_1\right] \quad (84)$$
$$+\left(\ln A_1\right)A_3A_1^{\left(x+A_2\right)^{A_3}}\left(A_3-1\right)\left(x+A_2\right)^{A_3-2}$$

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(1)}}{1-q_x^{(1)}}\right)= \quad (85)$$
$$\left(\ln A_1\right)^2\left(A_3\right)^2\left(x+A_2\right)^{2(A_3-1)}\left(A_1^{\left(x+A_2\right)^{A_3}}\right)+\left(\ln A_1\right)A_3A_1^{\left(x+A_2\right)^{A_3}}\left(A_3-1\right)\left(x+A_2\right)^{A_3-2}$$

$\dfrac{q_x^{(2)}}{1-q_x^{(2)}}$ requires that $x>0$ otherwise, it will undefined at perinatality. The function can be re-expressed as

$$\frac{q_x^{(2)}}{1-q_x^{(2)}}=B_1\exp\left(-B_2\left(\log_e\frac{x}{B_3}\right)^2\right) \quad (86)$$

The function (74) is maximum when $\log_e x=\log_e B_3\Rightarrow x=B_3$
And proportional to lognormal kernel but it is not lognormal density function. At perinatality $x\to 0^+$, $\log_e x=-\infty\Rightarrow\dfrac{q_x^{(2)}}{1-q_x^{(2)}}=0$. At advanced age $x=\infty$, $\dfrac{q_x^{(2)}}{1-q_x^{(2)}}=0$. As a result, in Figures 1 and 3, mortality rate declined to the age axis. The mortality slope is given as

$$\frac{d}{dx}\left(\frac{q_x^{(2)}}{1-q_x^{(2)}}\right)=B_1\frac{d}{dx}\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right) \quad (87)$$

$$\frac{d}{dx}\left(\frac{q_x^{(2)}}{1-q_x^{(2)}}\right)=\frac{2}{x}\left(-B_2\left(\log_e x-\log_e B_3\right)\right)B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right) \quad (88)$$

The function increases as $x<B_3$ and decreases as $x>B_3$ and the parameters have the following roles. B_1 describes the amplitude of the function, B_2 is the concentration and B_3 is the inverse variance in log age. The mortality curvature is given as

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(2)}}{1-q_x^{(2)}}\right)=\frac{d}{dx}\left\{\frac{2}{x}\left(-B_2\left(\log_e x-\log_e B_3\right)\right)B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\right\} \quad (89)$$

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(2)}}{1-q_x^{(2)}}\right)=\frac{d}{dx}\left\{\frac{2}{x}\left(-B_2\left(\log_e x-\log_e B_3\right)\right)B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\right\}=$$
$$\left\{\left(-B_2\left(\log_e x-\log_e B_3\right)\right)B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\frac{d}{dx}\frac{2}{x}\right\} \quad (90)$$
$$+\left\{\frac{2}{x}B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\frac{d}{dx}\left(-B_2\left(\log_e x-\log_e B_3\right)\right)\right\}$$
$$+\left\{\frac{2}{x}\left(-B_2\left(\log_e x-\log_e B_3\right)\right)B_1\frac{d}{dx}\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\right\}$$

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(2)}}{1-q_x^{(2)}}\right)=$$
$$\left\{\left(-B_2\left(\log_e x-\log_e B_3\right)\right)B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\left(\frac{-2}{x^2}\right)\right\}$$
$$+\left\{\frac{-2B_2}{x^2}B_1\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\right\} \tag{91}$$
$$+\left\{\frac{4\left(B_2B_1\right)^2}{x^2}\left(\log_e x-\log_e B_3\right)^2\left[\exp\left(-B_2\left(\log_e x-\log_e B_3\right)^2\right)\right]\right\}$$

The function (74) is purely the Gompertz's law of mortality.

$$\lim_{x\to 0}\left(\frac{q_x^{(3)}}{1-q_x^{(3)}}\right)=\lim_{x\to 0}C_1C_2^x=C_1 \tag{92}$$

$$\lim_{x\to \infty}\left(\frac{q_x^{(3)}}{1-q_x^{(3)}}\right)=\lim_{x\to \infty}C_1C_2^x=\infty;\quad C_2>1 \tag{93}$$

Following observations in Gavrilov and Gavrilova (2019a), mortality rate diverges exponentially for $C_2>1$ at old ages. The mortality slope and acceleration are respectively given as

$$\frac{d}{dx}\left(\frac{q_x^{(3)}}{1-q_x^{(3)}}\right)=\frac{d}{dx}C_1C_2^x=C_1\left(\ln C_2\right)C_2^x \tag{94}$$

$$\frac{d^2}{dx^2}\left(\frac{q_x^{(3)}}{1-q_x^{(3)}}\right)=C_1\left(\ln C_2\right)^2C_2^x \tag{95}$$

Survival data were extracted from German DAV 2008 and then normalized to obtain the results. The survival data was used to compute q_x and then equation (1) was implemented to estimate the parameters. In Table 1 for all the mortality functions, mortality rate intensity was generally very high during perinatality and then plummeted shortly after, the parameters of HP_1 are

$$HP_1\left(A_1,A_2,A_3,B_1,B_2,B_3,C_1,C_2:male\right)$$
$$=\begin{pmatrix}0.008859,0.080407,0.356156,0.000939587,7.012312,24.798101,\\0.00008187,1.08789\end{pmatrix} \tag{96}$$

$$HP_1\left(A_1,A_2,A_3,B_1,B_2,B_3,C_1,C_2:\ female\right)$$
$$=\begin{pmatrix}0.00808571,0.05017701,0.386176,0.000623876,10.714204,\\28.752123,0.0000649474,1.0857334\end{pmatrix} \tag{97}$$

The term A_3 is the force responsible for the rapid mortality rate decline throughout infancy. In Table 1 for $HP1$, the perinatal mortality was high at birth for male, declined sharply up to age 7 but seemed stable within $8\le x\le 9$. It then progressively increased until age 120 due to implicit or unpredictable shocks such as pandemics or war.

However, the female mortality sharply declined till age 9 after which mortality increased continuously till 120. For male, $\mu_x>1$ for $111\le x\le 120$ and for female, $\mu_x>1$ for

$118 \le x \le 120$. The trajectory of $HP1$ component-wise for both sexes was shown in Figures 1 and 3. Apparently, mortality rates at infancy, young adulthood and at adulthood sequentially preceded each other in a natural order, hence their joint mortality trend was presented through surface plots for both sexes in Figures 2 and 4. Since asymptotics is a long-run behaviour as x approaches infinity, it was obvious that in the first term, the individual mortality rate $\mu_x \to 0^+$ became significantly very low compared to 1. Because of the observed insignificantly smallness of this contribution to the overall mortality, it appeared to form asymptotes to both axes as seen in Figures 1 and 3. The second component is a lognormal function expressed as the accidental hump into higher adulthood within the interval $10 \le x \le 50$ for both sexes.

The lognormal behaviour caused its trajectories in Figures 1 and 3 to exhibit leptokurtic curves which described the shape of the mortality distribution in the second terms. The second component curves referred to in Figures 1 and 3 increased to a maximum around ages 25 respectively before falling back to the horizontal. From these Figures, the curves seemed to have short tails and more highly peaked when compared to a normal distribution. The implication was that the leptokurtic mortality distribution had a higher number of intensities clustered around the mean deaths with more extreme severities in the tails relative to a normal distribution. The third component curve described continuous asymptotes along the age axis up till age 60 before exponentially rising up. Although real mortality starts at 10, a close observation on all Heligman-Pollard's laws revealed that the following mortality conditions $\mu_x'(10) = 0$ and $\mu_x(0) = \int_0^1 \mu_x dx$ were not satisfied. The reason has to do with the lognormal behaviour in the second terms of the law.

Conclusion

The special function derivations for the analytic death severities constructed under the Heligman-Pollard's laws proved that the new mortality models were consistent with the theoretical expectations that the death rate severity function increased with age. The evolving hazard function governed the exponential decay of the survival function connecting the probabilistic and analytic representations of longevity and consequently, $\lim_{t\to\infty} \Lambda(t) = \infty$. The analytic invertibility of the survival probability through death severity function supported life annuities and insurance benefits.

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